Question: *Find the mean of a branching process with immigration (described in Question* 5.4.5*), in terms of the mean and variance of the offspring and immigrant distributions;*

Solution: Note first that the general solution to the recursion $a_{n+1} = ra_n + c$ is given by $a_n = (a_0 - a^*)r^n + a^*$, where a^*, the steady state solution, is defined by $a^* = ra^* + c$, so that $a^* = \frac{c}{1-r}$ (see e.g. Wikipedia). A simple way to see this is to divide through by $r^{-(n+1)}$, and solve the resulting recursion for $b_n = r^{-n}a_n$.

Let the mean of the offspring distribution be $\mu_o \neq 1$ and let the mean of the immigration distribution be μ_i. Then

$$\mathbb{E}[Z_{n+1}] = \mathbb{E}[\mathbb{E}[Z_{n+1}|Z_n]] = \mathbb{E}[\mu_o Z_n + \mu_i] = \mu_o \mathbb{E}[Z_n] + \mu_i.$$

You may also arrive at the recursion by differentiating the recursion for the generating function of Z_n. Solving the recursion,

$$\mathbb{E}[Z_n] = \mathbb{E}[Z_0]\mu_0^n + \mu_i \frac{\mu_o^n - 1}{\mu_o - 1}.$$

If $\mu_o = 1$, on the other hand, then we have that $\mathbb{E}[Z_n] = Z_0 + n\mu_i$.

Question:

(a) *Show, for any two linear fractional transformations*

$$f(s) = \frac{as+b}{cs+d} \quad \text{and} \quad g(s) = \frac{es+f}{gs+h}$$

that

$$f(g(s)) = \frac{As+B}{Cs+D},$$

where A, B, C *and* D *are defined by*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

(b) *Show that if* $\mathbb{P}(Y = 0) = d$ *and* $\mathbb{P}(Y = k) = (1-d)p(1-p)^{k-1}$ *for* $k > 0$, *then the generating function of* Y *has this form.*

(c) *Explain how to find the generating function of the branching process whose offspring distribution is* Y.

Solution: (a)

$$f(g(s)) = f\left(\frac{es+f}{gs+h}\right) = \frac{a\frac{es+f}{gs+h} + b}{c\frac{es+f}{gs+h} + d} = \frac{(ae+bg)s + (af+bh)}{(ce+dg)s + (cf+dh)}$$

(b)

$$\begin{aligned}
\mathbb{E}[s^Y] &= 1 \cdot \mathbb{P}(Y=0) + \sum_{k=1}^{\infty} s^k \mathbb{P}(Y=k) \\
&= d + ps(1-d) \sum_{k=1}^{\infty} s^{k-1}(1-p)^{k-1} \\
&= d + \frac{ps(1-d)}{1-s(1-p)} = \frac{s(p-d)+d}{s(p-1)+1}
\end{aligned}$$

(c) $G_{Z_n}(s) = \frac{A_n s + B_n}{C_n s + D_n}$ where

$$\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} = \begin{pmatrix} p-d & d \\ p-1 & 1 \end{pmatrix}^n$$

From this expression it is possible (although not necessary for the homework) to calculate A_n, B_n, C_n and D_n fairly easily. Although the matrix is not symmetric, it is (like all two-by-two matrices) similar to a diagonal matrix. Let $\mu = i\sqrt{d/(1-p)}$ (we may suppose that $p < 1$). Then, diagonalizing the matrix, if

$$V = \begin{pmatrix} \mu & 1 \\ -1/\mu & 1 \end{pmatrix},$$

and

$$\Lambda = \begin{pmatrix} 1-d & 0 \\ 0 & p \end{pmatrix},$$

we can check that

$$\begin{pmatrix} p-d & d \\ p-1 & 1 \end{pmatrix}^n = V\Lambda^n V^{-1}$$

Since Λ is diagonal, this is easy to compute.