Question:

(a) *Show that simple random walk on the infinite (rooted, regular) binary tree is transient by finding the probability that the walk ever returns to the root as a function of its starting depth.*

(b) *Call one subtree of the root the* right *subtree, and the other the* left*, and say that the level of a node is the number of steps needed to reach the root, multiplied by -1 if the node is in the left subtree. Show that the probability that the walk escapes to infinity out the right subtree is given by $r(k)$, where for $k \geq 0$,*

$$r(k) = \begin{cases} \frac{1}{2}\left(1 + (1 - 2^{-k})\right) & \text{for } k \geq 0 \\ \frac{1}{2}\left(1 - (1 - 2^{-k})\right) & \text{for } k < 0 \end{cases}$$

Solution: (a) Since we are only concerned with the depth of the current position, we may think of the simple random walk on the binary tree as a random walk $(X_n)_{n\geq 0}$ on $\{0, 1, 2, \ldots\}$ where $X_n = i$ means that we are at depth i at time n. The transition probabilities are given by $p_{0,1} = 1$ and

$$p_{i,i+1} = p \quad \text{and} \quad p_{i,i-1} = q = 1 - p$$

for $i \geq 1$, where $p = \frac{2}{3}$.

Fix some positive integer M. Let $T_{0,M}$ be the first time that the walk $(X_n)_{n\geq 0}$ hits 0 or M. For $0 \leq i \leq M$ let

$$f_M(i) = \mathbb{P}(X_{T_{0,M}} = 0 | X_0 = i).$$

That is, $f_M(k)$ is the probability that the walk starting at k hits 0 before M. We know that f_M is harmonic on $\{1, 2, \ldots, M-1\}$, namely (by conditioning on the first step) we have for $1 \leq i \leq M - 1$ that

$$f_M(i) = p_{i,i-1} f_M(i-1) + p_{i,i+1} f_M(i+1) = q f_M(i-1) + p f_M(i+1) \tag{1}$$

along with boundary conditions

$$f_M(0) = 1 \quad \text{and} \quad f_M(M) = 0 \tag{2}$$

Note: We could omit passing to the limit with M, by looking for a function f satisfying equation (1) for all $i > 0$ and having $f(0) = 1$ and $\lim_{i\to\infty} f(i) = 0$, but it pays to be careful. For instance, if we made M into a reflecting boundary (i.e. setting $p_{M,M-1} = 1$), then the random walk on $\{0, 1, \ldots, M\}$ would be recurrent, since any irreducible Markov chain on a finite state space is recurrent, and one might wonder if the function f we get is the correct one. In this case it works out, but by taking the limit explicitly, it is more clear what's going on.

The characteristic equation for the recurrence realtion (1) is $\lambda = q + p\lambda^2$, which has solutions $\lambda = 1$ and $\lambda = \frac{q}{p}$. The general solution to (1) is therefore given by

$$f_M(i) = A_M + B_M \left(\frac{q}{p}\right)^i$$

The boundary conditions in (2) imply that $A_M + B_M = 1$ and $A_M + B_M \left(\frac{q}{p}\right)^M = 0$. Thus

$$A_M = -\frac{q^M}{q^M - p^M} \quad \text{and} \quad B_M = \frac{p^M}{p^M - q^M}$$

Let us now take the limit as $M \to \infty$. Suppose, as is true in this case, that $p > q$. Then we have $\lim_{M\to\infty} A_M = 0$ and $\lim_{M\to\infty} B_M = 1$. Thus

$$\mathbb{P}(X \text{ hits zero eventually } | X_0 = i) = \lim_{M\to\infty} f_M(i) = \left(\frac{q}{p}\right)^i = 2^{-i} \qquad (3)$$

Now, if the probability that we ever return to zero, when starting at zero, is less than one, then the point zero will be transient, and hence the whole chain will be transient since it is irreducible. But since we always move to depth one when starting at zero, the probability that we ever return to zero, when starting at zero, is the same as the probability that starting at depth one we ever hit zero. By (3), this probability is 1/2. Hence the chain is transient.

If we define transience of zero as the number of returns to zero being almost surely finite, then we can give the following argument. The number of returns to zero, when starting at zero, is clearly equal to the number of returns to zero starting at depth 1, since our first move will always take us to depth 1. The probability that we ever hit zero starting at depth 1 is 1/2 by (3). If we hit zero we will return to depth 1, and once again we will make an eventual return to zero with probability 1/2. Thus the number of returns to zero, when starting at zero, is a geometric random variable with sucess parameter 1/2. This number is almost surely finite, and hence zero is a transient state for the chain $(X_n)_{n\geq 0}$. Since the chain $(X_n)_{n\geq 0}$ is irreducible, every state is transient.

(b) If we start at zero, then by symmetry the probability that we escape to $+\infty$ is the same as the probability we escape to $-\infty$; since these sum to one, the probablity of escaping to + beginning at zero is 1/2.

Then we know that if $f_M(k)$ is the probability that the walk hits $+M$ before hitting $-M$, that f_M is harmonic in $\{-M+1, \ldots, M-1\}$, namely, that

$$f_M(i) = \begin{cases} = qf_M(i-1) + pf_M(i+1) & \text{for } 1 \leq |i| \leq M-1 \\ = \frac{1}{2}(f_M(-1) + f_M(1)) & \text{for } i = 0 \end{cases}$$

and that f_M has boundary conditions $f_M(-M) = 0$ and $f_M(M) = 1$. This can be solved; and the task is made easier by the observation above that $f(0) = 1/2$, which we can add to our boundary conditions, and use the general form of the solution to equation (1) from part (a).

Alternatively, we can argue directly using part (a) as follows. If we start on the *right* side at depth k, then there are two ways we can excape to $+\infty$. Either we hit zero at some point, which happens with probability 2^{-k}, and then escape to $+\infty$ with probability $1/2$, or we never hit zero which happens with probability $1-2^{-k}$. Thus

$$r(k) = 2^{-k} \cdot \frac{1}{2} + (1 - 2^{-k}) = \frac{1}{2}\left(1 + (1 - 2^{-k})\right)$$

If we start on the *left* side at depth k, then there is only one way we can excape to $+\infty$. First we must hit zero, which happens with probability 2^{-k}, and then escape to $+\infty$ with probability $1/2$. Thus

$$r(-k) = 2^{-k} \cdot \frac{1}{2} = \frac{1}{2}\left(1 - (1 - 2^{-k})\right)$$

Question: *Let X be a Markov chain on $\{0, 1, 2, ...\}$ with transition probabilities given by*

$$P(k, k-1) = 1 \quad \text{and} \quad P(0, k) = P(Y = k)$$

for $k \geq 0$, and where Y is a random variable taking values in $\{1, 2, 3, ...\}$ with $\mathbb{E}[Y] < \infty$. Find the stationary distribution of X.

Solution:

The chain is clearly recurrent; it is positive recurrent if and only if $\mathbb{E}[Y] < \infty$.

The stationary distribution is

$$\pi_k = \frac{\mathbb{P}\{Y \geq k\}}{1 + \mathbb{E}[Y]},$$

as long as $\mathbb{E}[Y]$ is finite. Here is an easy way to see this. Let $\{T_1, T_2, \ldots\}$ be the times of a renewal process with inter-renewal intervals i.i.d. with distribution Y. Then if $\tau(k) = \min\{T_i : T_i \geq k\}$ is the time of the next renewal after time k, then $Z_k = \tau(k) - k$, the time until the next renewal, is a Markov chain with the same distribution as X, and $T_1 + 1$ corresponds to X_0. Since Z and T are equivalent, we know that making the renewal process T stationary will give us a stationary version of Z (and therefore X); and we have already found what the distribution of T_1 must be to make T stationary.

Here is another way to find this. Recall that if μ_x is the expected return time to x, beginning at x, and $\rho_y^{(x)}$ is the exepcted number of visits to y before hitting x after starting at x, then

$$\pi_y = \frac{\rho_y^{(x)}}{\mu_x}.$$

Take $x = 0$. It is easy to see that $\mu_x = \mathbb{E}[Y] + 1$, since if the chain jumps to k, it takes $k + 1$ steps to return. Furthermore, the chain visits each state at most once on each excursion from 0, and the probability it visits state k is $\mathbb{P}\{Y \geq k\}$; so $\rho_y^0 = \mathbb{P}\{Y \geq y\}$, and hence

$$\pi_y = \frac{\mathbb{P}\{Y \geq y\}}{\mathbb{E}[Y] + 1}.$$

A third approach, which is widely applicable, is to find π_0 and then use the relation $\pi P = P$ to find the rest of π. In this case, it is obvious that $\mu_0 = 1 + \mathbb{E}[Y]$, and hence $\pi_0 = 1/\mu_0 = \frac{1}{a+\mathbb{E}[Y]}$.