Question: *Let X be a nonnegative random variable with density $f(x)$ and cumulative distribution function $F(x)$. Let L be a Poisson point process on the nonnegative real line with intensity function*

$$g(x) = \frac{f(x)}{1 - F(x)},$$

and let $Y = \min\{y : y \in L\}$ be the smallest point in L. Show that Y has the same distribution as X.

Solution: Let $N(t)$ be the number of points of L in the interval $[0, t]$. By the standard properties of a Poisson point process, $N(t)$ has the Poisson distribution with mean

$$\int_0^t g(x)dx = \int_0^t \frac{f(x)}{1 - F(x)} dx$$

Since F'(t) = f(t), by the Fundamental Theorem of Calculus we get

$$\int_0^t g(x)dx = -\log(1 - F(x))|_{x=0}^{x=t} = -\log(1 - F(t))$$

where we have used the fact that $F(0) = \mathbb{P}(X \leq 0) = 0$, which is true since X is non-negative and hence $\mathbb{P}(X < 0) = 0$, and since $\mathbb{P}(X = 0) = 0$ because X has a density and is thus a continuous random variable. We will have $Y > t$ if and only if there are no points of L in the interval $[0, t]$. Thus

$$\mathbb{P}(Y > t) = \mathbb{P}(N(t) = 0) = e^{\log(1-F(t))} = 1 - F(t)$$

Hence if $f_Y(t)$ is the density of Y, and $F_Y(t)$ its cumulative distribution function, then

$$f_Y(t) = \frac{d}{dt} F_Y(t) = \frac{d}{dt}\mathbb{P}(Y \leq t) = \frac{d}{dt}(1 - (1 - F(t)) = f(t)$$

and thus Y has the same distribution as X. We don't even need to go this far, we could just say that as their cumulative distribution functions agree, they must have the same distribution.

Question: *A Poisson point process of raindrops of constant intensity u begins falling on the plane at time zero. Each raindrop splatters out to a circle of random radius; the radii are iid and exponentially distributed with mean 1. Find the probability that the origin has gotten wet by time t.*

Solution: We will use the coloring theorem (Theorem 6.13.14). We color any point of the Poisson point process red if the splash from the raindrop at that point gets the origin wet, and blue otherwise. Let Y be an exponential random variable with parameter 1. The probability that a point at a distance r from the origin is colored red is

$$\mathbb{P}(Y > r) = e^{-r}$$

By the coloring theorem, the red points form a Poisson process on $\mathbb{R}^2$ with intensity given in polar coordinates by $e^{-r}u$ per unit time. Thus per unit time, there is a Poisson number of points in each subset A of $\mathbb{R}^2$ with mean

$$\int_A e^{-r} u dx dy = \int_A r e^{-1} u dr d\theta$$

as long as this integral is finite. Taking $A = \mathbb{R}^2$, and thus finding the mean total number of red points per unit time, we have

$$u \int_{\mathbb{R}^2} r e^{-r} dr d\theta = u \int_0^{2\pi} \int_0^{\infty} r e^{-r} dr d\theta = 2\pi u \int_0^{\infty} r e^{-r} dr = 2\pi u$$

Thus the total number of red points by time t has the Poisson distribution with mean $2\pi t u$. Let $R(t)$ be the total number of red points by time t. The origin is not wet at time t if and only if there are no red points by time t, and the probability of this event is given by

$$\mathbb{P}\left(R(t) = 0\right) = e^{-2\pi t u}$$