Question: *Spiders live on a long telephone wire at the points of a Poisson point process with constant intensity 1; the spider at x has made a section of the wire sticky for the same distance $R(x)$ in both directions from its home at x; these distances $R(x)$ are iid and uniformly chosen on $[0, 1]$.*

(a) *Fix a length of wire L units long. Find the Laplace transform of the total amount of stickiness laid down by the spiders that live on that section. (i.e. the total area claimed by those spiders, counting overlapping areas once for each claiming spider)*

(b) *What is the probability that a given point on the wire is not sticky?*

(c) *A large number of flies land as a Poisson point process with constant intensity z on the wire. Denote by F_k the flies landing in sections claimed by exactly k spiders. What is the mean number of flies in F_k per unit length?*

Solution: **(a)** We will need the Laplace transform of the uniform distribution on $(0, 1)$ later in the question. Let $U \sim \text{Unif}(0, 1)$. Then

$$\mathbb{E}(e^{-sU}) = \int_0^1 e^{-su} du = s^{-1}(1 - e^{-s})$$

We could do this problem by directly applying Campbell's formula (check this gives the same answer!); we will take the somewhat more instructive route of conditioning on the number of spiders. Now, suppose there were k spiders in the interval of length L, at positions $x_1, \ldots, x_k$. Then the total amount of stickiness laid down by the spiders that live on that interval would be $\sum_{i=1}^{k} 2R(x_i)$, where $R(x_1), \ldots, R(x_k)$ are iid uniforms on $(0, 1)$. Note that this distribution does not depend on the exact positions $x_1, \ldots, x_k$.

Thus if X is the number of spiders in the interval of length L, and A is the total amount of stickiness laid down by those spiders, then

$$\mathbb{E}(e^{-sA} | X = k) = \mathbb{E}(e^{-2s \sum_{i=1}^{k} R(x_i)})$$

First by the independence of the $R(x_i)$'s, and then since they are identically distributed, we have

$$\mathbb{E}(e^{-sA} | X = k) = \prod_{i=1}^{k} \mathbb{E}(e^{-2sR(x_i)}) = \mathbb{E}(e^{-2sR(x_1)})^k$$

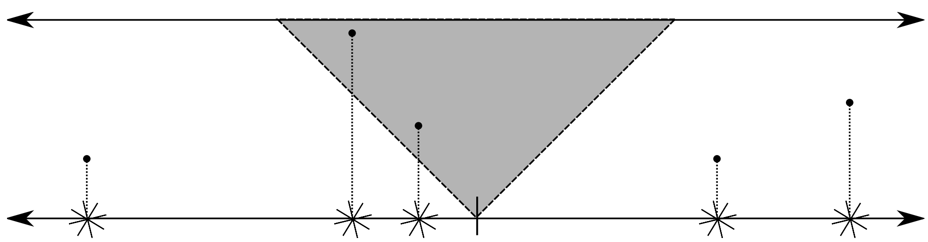
By evaluating the Laplace transform of the uniform distribution at $2s$ we get

$$\mathbb{E}(e^{-sA} | X = k) = \left((2s)^{-1}(1 - e^{-2s})\right)^k = \frac{1}{2^k s^k}(1 - e^{2s})^k$$

Now we need to find the Laplace transform of A but unconditionally on X:

$$\begin{aligned}\mathbb{E}(e^{-sA}) &= \sum_{k=0}^{\infty} \mathbb{E}(e^{-sA}|X=k)\mathbb{P}(X=k) \\ &= \sum_{k=0}^{\infty} \frac{1}{2^k s^k}(1-e^{-2s})^k \cdot \frac{L^k e^{-L}}{k!} \\ &= \exp\left\{ L\left(\frac{1}{2s}(1-e^{-2s}) - 1\right)\right\}\end{aligned}$$

(b) If R_i is the radius occupied by the spider living at point x_i, then by the marking theorem, the points $\{(x_i, R_i)\}$ form a Poisson point process on $\mathbb{R}\times[0,1]$ with constant intensity 1.



The spiders who have claimed the origin are those represented by points in $\{(x_i, R_i) : |x_i| \le R_i\}$; i.e. the shaded triangle in the figure. This triangle has area 1; since the intensity of the (two-dimensional) Poisson process is 1, the number of points inside it is Poisson with mean 1. Therefore, the probability that no spiders get the origin sticky is e^{-1}.

This could also be done with the marking theorem, as in the previous homework.

(c) By the arguments given in part (b), the number of spiders that get any particular point sticky has the Poisson distribution with mean 1. Thus the probability that a particular point is made sticky by exactly k spiders is $\frac{e^{-1}}{k!}$. (In the figure, there are two such spiders.) Fix a length of wire L units long; N flies land on it, where N is Poisson distributed with mean zL. Each fly has probability $\frac{e^{-1}}{k!}$ of landing on a spot claimed by k spiders; let $Y_j = 1$ if the jth fly does so, and $Y_j = 0$ otherwise. Then $F_k = \{1 \le j \le N : Y_j = 1\}$. The Y are *not* independent, but by linearity, and since they are each independent of N, $\mathbb{E}[|F_k|] = \mathbb{E}[\sum Y_j | N] = \frac{e^{-1}}{k!}\mathbb{E}[N] = \frac{e^{-1}}{k!} zL$. Therefore, the mean number of such flies per unit length is

$$\frac{\mathbb{E}[|F_k|]}{L} = \frac{e^{-1}}{k!} z.$$