Question: *Let A_1, B_1, A_2, B_2 be uncorrelated RVs with mean zero and variance 1, let u_1 and u_2 be distinct nonnegative numbers, and let L be a RV with $P(L = 1) = p$ and $P(L = 2) = 1 - p$, independently of the As and Bs. For each integer n, let*

$$X_n = p(A_1 \cos(nu_1) + B_1 \sin(nu_1)) + (1-p)(A_2 \cos(nu_2) + B_2 \sin(nu_2)),$$

$$Y_n = A_L \cos(nu_L) + B_L \sin(nu_L).$$

Show that X and Y are stationary, find their spectral distributions, and describe their spectral representations, in the sense of Theorem 9.4(4).

Without loss of generality, we take $0 < u_1 < u_2$.

Solution: (a)

$$\mathbb{E}(X_n) = p(\mathbb{E}(A_1)\cos(nu_1) + \mathbb{E}(B_1)\sin(nu_1)) + (1-p)(\mathbb{E}(A_2)\cos(nu_2) + \mathbb{E}(B_2)\sin(nu_2)) = 0$$

By bilinearity of covariance and the fact that A_1, B_1, A_2, B_2 are uncorrelated and hence have zero covariance,

$$\mathrm{Cov}(X_n, X_{n+m}) = p^2\mathrm{Cov}(A_1, A_1)\cos(nu_1)\cos((n+m)u_1) + p^2\mathrm{Cov}(B_1, B_1)\sin(nu_1)\sin((n+m)u_1)$$

$$+(1-p)^2\mathrm{Cov}(A_2, A_2)\cos(nu_2)\cos((n+m)u_2) + (1-p)^2\mathrm{Cov}(B_2, B_2)\sin(nu_2)\sin((n+m)u_2)$$

Since $\mathrm{Cov}(Z, Z) = \mathrm{Var}(Z) = 1$ for $Z = A_1, B_1, A_2, B_2$, we get

$$\begin{aligned}\mathrm{Cov}(X_n, X_{n+m}) &= p^2 \cos(nu_1)\cos((n+m)u_1) + p^2 \sin(nu_1)\sin((n+m)u_1) \\ &\quad +(1-p)^2 \cos(nu_2)\cos((n+m)u_2) + (1-p)^2 \sin(nu_2)\sin((n+m)u_2) \\ &= p^2 \cos(mu_1) + (1-p)^2 \cos(mu_2)\end{aligned}$$

where we have used the identities $\cos(a)\cos(b) = \frac{1}{2}(\cos(a-b) + \cos(a+b))$ and $\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))$. Thus X is weakly stationary since $\mathrm{Cov}(X_n, X_{n+m})$ depends only on m. Now we find the autocorrelation function.

$$\begin{aligned}\rho(m) &= \frac{\mathrm{Cov}(X_n, X_{n+m})}{\sqrt{\mathrm{Var}(X_n)\mathrm{Var}(X_{n+m})}} \\ &= \frac{1}{p^2 + (1-p)^2}(p^2 \cos(mu_1) + (1-p)^2 \cos(mu_2)) \qquad (1)\end{aligned}$$

Recall that the spectral distribution is the cumulative distribution function F such that

$$\rho(m) = \int_{-\pi}^{\pi} e^{im\lambda} dF(\lambda)$$

We almost have this already, since we have ρ as a sum of cosine functions. (Warning: There are different ways of writing the spectral distribution, corresponding to equations 9.3.12 and 9.3.14 in G & S; even though the process is real-valued, we need the first one, to match with Theorem 9.4.4.) Using the fact that $\cos(a) = \frac{1}{2}(e^{ia} + e^{-ia})$, equation (1) is

$$\rho(m) = \frac{1}{2(p^2 + (1-p)^2)} (p^2(e^{imu_1} - e^{-imu_1}) + (1-p)^2(e^{imu_2} - e^{-imu_2})) \quad (2)$$

and we have written the autocorrelation function ρ as a sum of sinusoids with different amplitudes and frequencies, which was our goal. Therefore, if we let $q = p^2(p^2 + (1-p)^2)$, we want the distribution F to assign mass $q/2$ to the two points u_1 and $-u_1$, and mass $(1-q)/2$ to the two points u_2 and $-u_2$, so

$$F(\lambda) = \begin{cases} 0 & \text{if } \lambda < -u_2, \\ \frac{(1-p)^2}{2(p^2+(1-p)^2)} & \text{if } -u_2 \le \lambda \le -u_1, \\ 1/2 & \text{if } -u_1 \le \lambda \le u_1, \\ 1 - \frac{p^2}{2(p^2+(1-p)^2)} & \text{if } u_1 \le \lambda \le u_2, \\ 1 & \text{if } \lambda \ge u_2. \end{cases}$$

Now we describe the spectral representation. Recall that the spectral representation of a discrete time process X is a complex valued stochastic process $S = \{S(\lambda) : -\pi \le \lambda \le \pi\}$ such that

$$X_n = \int_{-\pi}^{\pi} e^{in\lambda} dS(\lambda),$$

and that $\mathbb{E}[(S(u) - S(v))^2] = F(u) - F(v)$. This last fact implies that dS can only be nonzero where F assigns mass, i.e. F is nonconstant. Therefore, we know that S can have jumps only at $\pm u_1$, $\pm u_2$. If we let the sizes of these jumps equal $J_{\pm 1}$ and $J_{\pm 2}$ respectively, then

$$X_n = J_{-1}e^{-inu_1} + J_{+1}e^{+inu_1} + J_{-2}e^{-inu_2} + J_{+2}e^{+inu_2}.$$

Again using that $\cos(a) = \frac{1}{2}(e^{ia} + e^{-ia})$ and $\sin(a) = \frac{-i}{2}(e^{ia} - e^{-ia})$, we can rewrite the definition of X_n to read

$$X_n = \frac{1}{2}p(A_1 - iB_1)e^{-inu_1} + \frac{1}{2}p(A_1 + iB_1)e^{inu_1} + \frac{1}{2}(1-p)(A_2 - iB_2)e^{-inu_2} + \frac{1}{2}(1-p)(A_2 + iB_2)e^{inu_2}$$

If we equate these two expressions, we get that

$$J_{-1} = \frac{1}{2}p(A_1 - iB_1) \quad J_1 = \frac{1}{2}p(A_1 + iB_1)$$

and

$$J_{-2} = \frac{1}{2}(1-p)(A_2 - iB_2) \quad J_2 = \frac{1}{2}(1-p)(A_2 + iB_2).$$

In summary, $S(\lambda)$ has jumps of random size J_{-1}, J_1, J_{-2}, J_2 at $\lambda = -u_1, u_1, -u_2, u_2$ respectively, and is flat elsewhere.

(b) Now X is a *random choice of (random) sine wave*, rather than a *sum of (random) sine waves*. By conditioning on L we have

$$\mathbb{E}(Y_n) = p(\mathbb{E}(A_1)\cos(nu_1)+\mathbb{E}(B_1)\sin(nu_1))+(1-p)(\mathbb{E}(A_2)\cos(nu_2)+\mathbb{E}(B_2)\sin(nu_2)) = 0$$

Now,

$$\mathrm{Cov}(Y_n, Y_{n+m}) = \mathbb{E}(Y_n Y_{n+m}) - \mathbb{E}(Y_n)\mathbb{E}(Y_{n+m}) = \mathbb{E}(Y_n Y_{n+m}) = \mathbb{E}[\mathbb{E}(Y_n Y_{n+m}|L])$$

which after a little computation reveals that

$$\mathrm{Cov}(Y_n, Y_{n+m}) = p\cos(mu_1) + (1-p)\cos(mu_2).$$

The variance is now $p + (1-p) = 1$, so this is also the autocorrelation function. We can now proceed as in part (a), getting that the distribution F puts mass at $\pm u_1$ and $\pm u_2$, except that the masses are of size $p/2$ and $(1-p)/2$.

The spectral represenation is slightly different, but we can find it in the same way. As before, let J_{-1}, J_1, J_{-2}, J_2 be the jumps of $S(\lambda)$ at $\lambda = -u_1, u_1, -u_2, u_2$ respectively. Then we know that if $L = 1$, that J_{-2} and J_2 are zero, and that if $L = 2$, that J_{-1} and J_1 are zero. So, again working as in (a), we find that

$$J_{-1} = \frac{1}{2}(A_1 - iB_1) \quad J_1 = \frac{1}{2}(A_1 + iB_1) \quad \text{if } L = 1,$$

and that

$$J_{-2} = \frac{1}{2}(A_2 - iB_2) \quad J_2 = \frac{1}{2}(A_2 + iB_2) \quad \text{if } L = 2.$$

Question: *Let $X(t)$ be a Gaussian process on $[0, \infty)$ (not stationary!) with mean zero, and covariance function*

$$Cov(X(s), X(t)) = \min(s, t))$$

for all $t, s \geq 0$. For p between 0 and 1, find the distribution of

$$Z = X(t + ps) - (1 - p)X(t) - pX(t + s).$$

Show that Z is independent of $X(t)$ and $X(t + s)$, for all p.

Solution: Recall that if $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ then

$$W := aX + bY \sim \mathcal{N}(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\text{Cov}(X, Y))$$

Since $X(t) \sim \mathcal{N}(0, t)$, $X(t + s) \sim \mathcal{N}(0, t + s)$ and $X(t + ps) \sim \mathcal{N}(0, t + ps)$,

$$\begin{aligned}
Z &= X(t + ps) - (1 - p)X(t) - pX(t + s) \\
&\sim \mathcal{N}(0 - (1 - p) \cdot 0 - p \cdot 0, (t + ps) + (1 - p)^2 t + p^2(t + s) \\
&\quad -2(1 - p)\text{Cov}(X(t + ps), X(t)) - 2p\text{Cov}(X(t + ps), X(t + s)) + 2p(1 - p)\text{Cov}(X(t), X(t + s))) \\
&= \mathcal{N}(0, (t + ps) + (1 - p)^2 t + p^2(t + s) - 2(1 - p)t - 2p(t + ps) + 2p(1 - p)t) \\
&= \mathcal{N}(0, p(1 - p))
\end{aligned}$$

Since Z, $X(t)$ and $X(t + s)$ are all normally distributed, to show that Z is independent of $X(t)$ and $X(t + s)$ it is enough to show that

$$\text{Cov}(Z, X(t)) = \text{Cov}(Z, X(t + s)) = 0$$

Now,

$$\begin{aligned}
\text{Cov}(Z, X(t)) &= \text{Cov}(X(t + ps) - (1 - p)X(t) - pX(t + s), X(t)) \\
&= \text{Cov}(X(t + ps), X(t)) - (1 - p)\text{Cov}(X(t), X(t)) - p\text{Cov}(X(t + s), X(t)) \\
&= t - (1 - p)t - pt \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\text{Cov}(Z, X(t + s)) &= \text{Cov}(X(t + ps) - (1 - p)X(t) - pX(t + s), X(t + s)) \\
&= \text{Cov}(X(t + ps), X(t + s)) - (1 - p)\text{Cov}(X(t), X(t + s)) - p\text{Cov}(X(t + s), X(t + s)) \\
&= (t + ps) - (1 - p)t - p(t + s) \\
&= 0
\end{aligned}$$

By the way, X is also known as "Brownian motion," which we will encounter again.