Question: *Let X be a CTMC on $\{1, 2, 3, ...\}$ with generator matrix $G_{n,n+1} = nb$ and $G_{n,n} = -nb$ ('everyone gives birth independently at rate b'). Show that*

$$p_{1,n}(t) := \mathbb{P}(X(t) = n | X(0) = 1) = e{-}bt(1 - e^{-bt})^{n-1}$$

by verifying that this solves the forwards equation. Then write down $p_{m,n}(t) := \mathbb{P}(X(t) = n | X(0) = m)$ using the observation that this is a geometric distribution.

Solution: The forward equation is $P_t' = P_t G$. Using this, we see that the equations that $p_{1,n}(t)$ must satisfy are

$$p_{1,1}'(t) = -bp_{1,1}(t) \quad \text{and} \quad p_{1,n}'(t) = b(n-1)p_{1,n-1}(t) - bnp_{1,n}(t) \text{ for } n \geq 2.$$

It is easy to check that $p_{1,n}(t)$ as given in the question satisfies these equations since

$$\frac{d}{dt}\left(e^{-bt}(1 - e^{-bt})^{n-1}\right) = be^{-bt}(1-e^{-bt})^{n-2}(ne^{-bt}-1) = be^{-bt}(1-e^{-bt})^{n-2}((n-1)-n(1-e^{-bt}))$$

Thus the distribution of $X(t)$ given that $X(0) = 1$ is geometric with parameter e^{-bt}. Thinking of the process as a birth process, we know that each individual alive a time zero generates an independent geometric number of people by time t. Thus given $X(0) = m$, we have that $X(t)$ is the sum of m independent geometric distributions, each with paramter e^{-bt}. This is just then the negative binomial distribution with parameters (m, e^{-bt}) (where we are including 'successes' in our negative binomial count, so that it is a distribution on $\{m, m+1, \ldots\}$ rather than $\{0, 1, \ldots\}$).

The density is thus given by

$$p_{m,n}(t) = \binom{(n-m)+m-1}{n-m}(e^{-bt})^m(1-e^{-bt})^{n-m} = \binom{n-1}{n-m}e^{-mbt}(1-e^{-bt})^{n-m}$$

for $n \geq m$ and $p_{m,n}(t) = 0$ otherwise.