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(Based on notes from Sept. 2006)
Physics 351

Dimensional Analysis

UNITS AND DIMENSIONS

All physical quantities have units with which they are measured – kilograms, acres, Newtons, etc. Some units are, somewhat arbitrarily, "fundamental," e.g. the units of mass, length, and time. Other "derived" units can be expressed in terms of these. For example, the SI units *kg*, *m*, and *s* (kilograms, meters, and seconds) are fundamental, representing the physical quantities mass (M), length (L), and time (T). The unit of force, the Newton, is a derived unit: $1\ N = 1\ kg\ m\ /\ s^2$. The quantities L, M, and T are a useful (though not unique) set of properties on which to base our fundamental units.

The **dimension** of a physical quantity, A, often written $[A]$, is the function of the fundamental set of properties that corresponds to the units of A. Despite the complexity of the preceding sentence, the dimension is easy to understand. The dimensions of force, for example, are $[F] = M\,L\ /\ T^2$. It is, of course, important that any physical relation, e.g. an equation, have the same dimensions "on each side." Mass must equal mass, for example, not mass × length. (On your problem sets, we'll take off more points for dimensionally incorrect answers than for numerically incorrect answers!) More interestingly, dimensional relations can be used to *construct* correct physical relationships...

DIMENSIONAL ANALYSIS

... a skill that is very useful in physics and other sciences, and which is rarely taught. The preliminary step is to figure out what the dimensions of a quantity are. Any (correct) expression can guide you. Let's consider energy, E. What is $[E]$? Recall that kinetic energy $Ek = (½)\ m\ v^2$, where m is mass and v is velocity. Therefore $[E] = M\,L^2\,T^{-2}$, since $[m] = M$ and $[v] = L/T$. Note that numerical factors, e.g. ½, are irrelevant to the dimensions.

The essence of dimensional analysis is relating the dimensions of the quantity of interest to the dimensions of the parameters it might depend on. I'll explain this with some examples:

(1) What is the period of a pendulum?

Consider a mass *m* hanging at the end of a pendulum string of length ℓ. What is its period of oscillation? Of course, this is very relevant to this course, and of course, we all know the answer from first-year mechanics, but let's harness dimensional analysis.

The period, τ, has dimensions of time, *T*. Without knowing anything we might suppose that τ can depend, let's say, on ℓ, *m*, and the gravitational acceleration, *g*. So $[\tau]$ = *function of* ($[\ell]$, $[m]$, $[g]$), where "*function of*" is just some algebraic expression – products of powers of the arguments:

$[\tau] = [\ell]^a, [m]^b, [g]^c$, where a, b, and c are exponents we need to determine.

The dimensions $[\ell] = L$, $[m] = M$, $[g] = LT^{-2}$, and $[\tau] = T$. So the above expression becomes

$T = L^a, M^b, (LT^{-2})^c$.

There is no "*M*" on the left side of this equation, so b must be equal to zero. (This is our first interesting observation.) The only a and c that will work (think about this) are a = +½ and c = -½.

Therefore, $[\tau] = \left[\sqrt{l/g}\right]$ and

τ = (dimensionless constant) $\sqrt{(\ell / g)}$.

Aside from a numerical constant (which happens to be 2π, by the way), dimensional analysis has given us the correct expression for the period of a pendulum, and even told us that the pendulum mass doesn't matter! Of course, we had to think a little bit at the beginning, supplying a physically reasonable guess for the parameters on which τ depends.

(2) How energetic is an atomic bomb blast, and how do you determine this, if the answer is a secret?

Next, a less trivial example. Following the development of atomic weapons in the 1940's, the energy, *E*, released by an atomic bomb blast was maintained as classified, secret information. *Life* magazine published photos of the blasts, showing the expanding fireball (of radius *r*) at a series of times (*t*) following detonation (see Figure, page 4). From these publicly available photos, G. I. Taylor, a very clever physicist, figured out the value of *E*. Here's how.

Taylor reasoned that *r*, the blast radius, should depend on *t* (of course), *E*, and the density of air (ρ). It might depend on other parameters, such as the atmospheric pressure, but for reasons we'll just briefly touch on, Taylor concluded that these are irrelevant. The blast pressure, for example, greatly exceeds the ambient atmospheric pressure, and so the atmospheric pressure can't play a significant role in the blast's properties. The task now is simply to find by dimensional analysis an expression relating *r*, *t*, *E*, and ρ:

$[r] = [E]^a [\rho]^b [t]^c$, or $r = (\text{const.})\, E^a \rho^b t^c$.

The dimensions $[r] = L$, $[E] = M L^2 T^{-2}$, $[\rho] = ML^{-3}$ (mass / volume), and $[t] = T$. Therefore, our dimensional relation becomes:

$$\begin{aligned} L &= (M L^2 T^{-2})^a (ML^{-3})^b T^c \\ &= M^{(a+b)} L^{(2a-3b)} T^{(-2a+c)} \end{aligned}$$

Examining this:

- There are no "*M*"s on the left side, so a+b = 0; i.e. a = -b.
- There are no "*T*"s on the left side, so -2a+c = 0, i.e. c = 2a.
- There is 1 power of *L* on the left side, so 2a-3b = 1.

Therefore (combining lines 3 and 1), 5a = 1, i.e. a = 1/5, b = -1/5, and c = 2/5.

Therefore: $r(t) = \left(\frac{Et^2}{\rho}\right)^{\frac{1}{5}}$ × (dimensionless constant).

Given the density of air, one can read values of *r* and *t* from the photos and *thereby extract E* ! The best way to do this is to plot log(*r*) vs. log(*t*) (see Figure, page 4). The slope "should," from the above expression, be 2/5 – measuring it provides verification that the dimensional analysis approach, and the choice of dependent parameters, is reasonable. The intercept of the log-log plot gives *E*. (Recall that log(*AB*) = log(*A*) + log(*B*), and log(A^x) = *x*log(*A*).)

Note that *dimensional analysis cannot reveal anything about "dimensionless constants"* in physical expressions. Typically, however, these constants are things like 2π, etc. – numbers "of order 1."

FURTHER READING:

On dimensional analysis and related issues, see
Scaling by G. I. Barenblatt, Cambridge Univ. Press, Cambridge, 2003.

On G. I. Taylor see, e.g.,
Modern classical physics through the work of G. I. Taylor, M. P. Brenner & H. A. Stone, *Physics Today*, May 2000.

Figure: *Next Page*

From Barenblatt (2003) (see notes, p. 6)

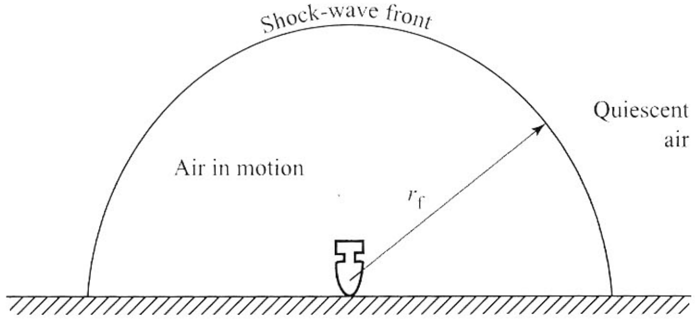

Figure 0.1. A very intense shock wave propagating in quiescent air.

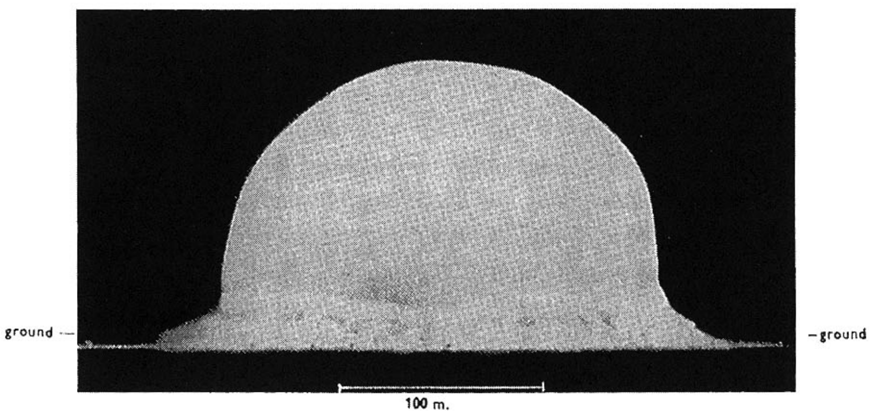

Figure 0.2. Photograph of the fireball of the atomic explosion in New Mexico at $t = 15$ ms, confirming in general the spherical symmetry of the gas motion (Taylor 1950b, 1963).

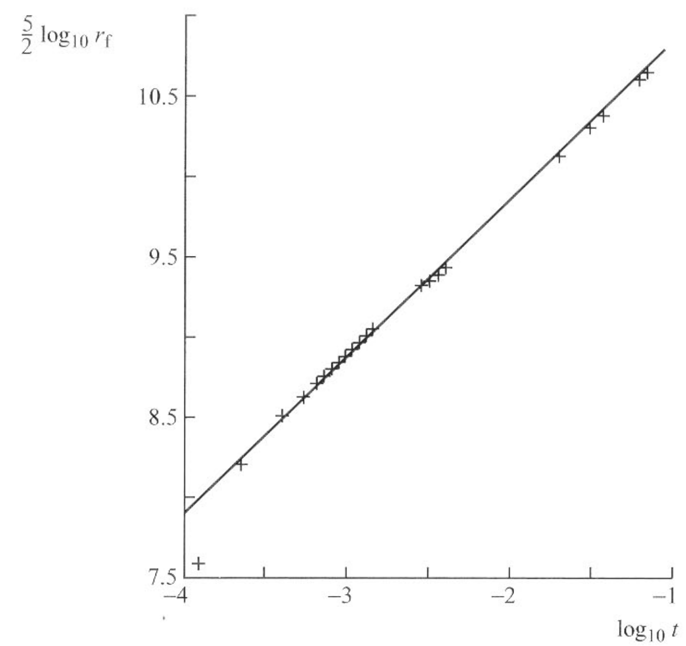

Figure 0.3. Logarithmic plot of the fireball radius, showing that $r_f^{5/2}$ is proportional to the time t (Taylor 1950b, 1963).