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Physics 351 – Vibrations and Waves

Problem Set 1: SOLUTIONS

1.a. the motion equation is

$m\ddot{x} = -kx \qquad \omega = \sqrt{k/m}$

that is $\ddot{x} + \omega^2 x = 0$

if $x = A\sin\omega t + B\cos\omega t$.

then $\dot{x} = A\omega\cos\omega t - B\omega\sin\omega t$.

so $\ddot{x} = -A\omega^2\sin\omega t - B\omega^2\cos\omega t$

$= -\omega^2(A\sin\omega t + B\cos\omega t)$

$\therefore \ddot{x} + \omega^2 x = (A\sin\omega t + B\cos\omega t)(\omega^2 - \omega^2) = 0$.

so $x(t) = A\sin(\omega t) + B\cos(\omega t)$ is a solution to the equation.

1.b. if $x(t) = C\sin(\omega t - \varphi)$

then $\dot{x}(t) = \omega C\cos(\omega t - \varphi)$

$\ddot{x}(t) = -\omega^2 C\sin(\omega t - \varphi)$

so $\ddot{x}(t) + \omega^2 x = -\omega^2 C\sin(\omega t - \varphi) + \omega^2 C\sin(\omega t - \varphi) = 0$

$\therefore x(t) = C\sin(\omega t - \varphi)$ is also a solution.

1.c. if $x(t) = D\sin\omega t + E\cos(\omega t - F)$

then $\dot{x}(t) = D\omega\cos\omega t - E\omega\sin(\omega t - F)$

$\ddot{x}(t) = -\omega^2 D\sin\omega t - \omega^2 E\cos(\omega t - F)$

$= -\omega^2\left(D\sin\omega t + E\cos(\omega t - F)\right)$

$= -\omega^2 x(t)$

so $\ddot{x}(t) + \omega^2 x(t) = -\omega^2 x(t) + \omega^2 x(t) = 0$

so his $x(t)$ is also a solution to the differential equation.

we have $x(t) = D\sin\omega t + E\cos(\omega t - F)$

$= D\sin\omega t + E(\cos\omega t\cos F + \sin\omega t\sin F)$

$= D\sin\omega t + E\cos F\cos\omega t + E\sin F\sin\omega t$

$= (D + E\sin F)\sin\omega t + E\cos F\cos\omega t$

if we set. $A = D + E\sin F$

$B = E\cos F$

then $x(t) = A\sin\omega t + B\cos\omega t$

so we only need two arbitrary parameters.

2. a. $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x) x^n$

So $\sin x = \frac{1}{0!}\sin(0)x^0 + \frac{1}{1!}\cos(0)x^1 + \frac{1}{2!}(-\sin x)|_{x=0}\, x^2$

$+ \frac{1}{3!}(-\cos x)|_{x=0}\, x^3 + \frac{1}{4!}(+\sin x)|_{x=0}\, x^4 + \frac{1}{5!}(\cos x)|_{x=0}\, x^5 + o(1)$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(1)$

b. $y(x) = \frac{1}{\sqrt{x}} + x$

$y'(x) = -\frac{1}{2}\frac{1}{x\sqrt{x}} + 1 = 0$

$\therefore x^{\frac{3}{2}} = \frac{1}{2}$

$x = \sqrt[3]{\frac{1}{4}}$

when $0 < x < \sqrt[3]{\frac{1}{4}}$ $y'(x) < 0$.

when $x > \sqrt[3]{\frac{1}{4}}$ $y'(x) > 0$

so at $x_0 = \sqrt[3]{\frac{1}{4}}$ $y(x)$ is minimal.

$y''(x) = -\frac{1}{2}\times(-\frac{3}{2})x^{-\frac{5}{2}} = \frac{3}{4}x^{-\frac{5}{2}}$

$y^{(3)}(x) = \frac{3}{4}\times(-\frac{5}{2})x^{-\frac{7}{2}} = -\frac{15}{8}x^{-\frac{7}{2}}$

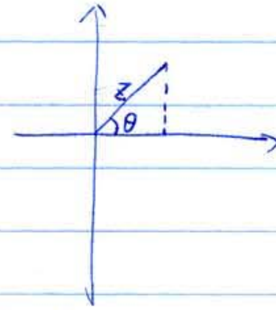
$\therefore\ y''(x_0) = \frac{3}{4}\times(\frac{1}{4})^{\frac{1}{3}\times -\frac{5}{2}} = \frac{3}{4}\times(\frac{1}{4})^{-\frac{5}{6}} = 3\times(\frac{1}{4})^{\frac{1}{6}} = 3\times 4^{-\frac{1}{6}}$

$y^{(3)}(x_0) = -\frac{15}{8}\times(\frac{1}{4})^{\frac{1}{3}\times(-\frac{7}{2})} = -\frac{15}{8}\times(\frac{1}{4})^{-\frac{7}{6}} = -\frac{15}{2}\times(\frac{1}{4})^{-\frac{1}{6}} = -\frac{15}{2}\times 4^{\frac{1}{6}}$

So $y(x) = y(x_0) + \frac{1}{1!}y'(x_0)(x-x_0) + \frac{1}{2!}y''(x_0)(x-x_0)^2 + \frac{1}{3!}y^{(3)}(x_0)(x-x_0)^3 + o(1)$

$= 4^{\frac{1}{6}} + 4^{-\frac{1}{3}} + \frac{1}{2}\times 3\times 4^{-\frac{1}{6}}(x-(\frac{1}{4})^{\frac{1}{3}})^2 + \frac{1}{6}\times(-\frac{15}{2})\times 4^{\frac{1}{6}}(x-(\frac{1}{4})^{\frac{1}{3}})^3 + o(1)$

3. $e^{j\theta} = \cos\theta + j\sin\theta$



$z e^{j\theta} = z\cos\theta + j z\sin\theta$

$\therefore |z e^{j\theta}| = \left(z e^{j\theta} \cdot (z e^{j\theta})^*\right)^{\frac{1}{2}}$

$= \left(z \cdot z^* \cdot e^{j\theta} \cdot e^{-j\theta}\right)^{\frac{1}{2}}$

$= |z|$

so there is no alteration of the length.

if $z = a + jb$.

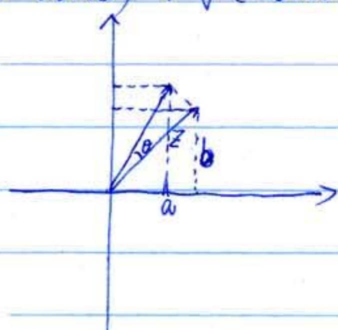
the pole angle of z is ϕ

the $\tan\phi = \frac{b}{a}$.

$z e^{j\theta} = (a+jb)(\cos\theta + j\sin\theta)$

$= a\cos\theta + a\sin\theta j + b\cos\theta j - b\sin\theta$

$= (a\cos\theta - b\sin\theta) + j(a\sin\theta + b\cos\theta)$



set the new angle as ϕ'

then $\tan\phi' = \dfrac{a\sin\theta + b\cos\theta}{a\cos\theta - b\sin\theta} = \dfrac{\left(\frac{a\sin\theta}{b\cos\theta} + 1\right) b\cos\theta}{\left(1 - \frac{b\sin\theta}{a\cos\theta}\right) a\cos\theta}$

$= \dfrac{\frac{\tan\theta}{\tan\phi} + 1}{1 - \tan\theta\tan\phi} \cdot \tan\phi$

$= \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \tan(\theta + \phi)$

$\therefore \phi' = \theta + \phi$ so we show that multiplication of any complex number z by $e^{j\theta}$ can be described as a positive rotation by θ.

4. $\cos jx = \cos x + j\sin x$

so $\cos x = \frac{1}{2}(e^{jx} + e^{-jx})$

$\sin x = \frac{1}{2j}(e^{jx} - e^{-jx})$

so $\sin(u \pm v) = \frac{1}{2j}\left(e^{j(u\pm v)} - e^{-j(u\pm v)}\right)$

$= \frac{1}{2j}\left(e^{ju}\cdot e^{\pm jv} - e^{-ju}\cdot e^{\mp jv}\right)$

$= \frac{1}{2j}\left((\cos u + j\sin u)(\cos v \pm j\sin v) - (\cos u - j\sin u)(\cos v \mp j\sin v)\right)$

$= \frac{1}{2j}\Big(\cos u\cos v + j\sin u\cos v \pm j\cos u\sin v \mp \sin u\sin v$

$\quad - [\cos u\cos v - j\sin u\cos v \pm j\cos u\sin v \mp \sin u\sin v]\Big)$

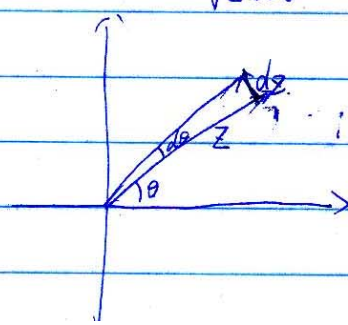
$= \sin u\cos v \pm \cos u\sin v$

5. $z = Ae^{j\theta}$

then $dz = A\,de^{j\theta}$

$= Aje^{j\theta}\,d\theta$

$= jz\,d\theta$



$\therefore$ the direction of dz is perpendicular to that of vector $(\mathrm{Re}\,z, \mathrm{Im}\,z)$.

$dz = z\,d\theta\,e^{j\frac{\pi}{2}}$

$= jz\,d\theta$

6

We want to know how much $\$i^i$ is worth. We start by rearranging our given expression with the identity,

$$i = e^{\ln i}.$$

So we have

$$i^i = (e^{\ln i})^i = e^{i\ln i}.$$

Notice also that

$$i\frac{\pi}{2} = \ln(e^{\frac{i\pi}{2}}) = \ln(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = \ln i.$$

Plugging this result into the previous equation gives us

$$i^i = e^{i\ln i} = e^{ii\frac{\pi}{2}} = e^{-\frac{\pi}{2}} = 0.21.$$

So $\$i^i$ is 21 cents. It's worth it !

Comment: Think about what $e^{i\pi/2}$ looks like graphically, in the complex plane – a 90° rotation!

7. a. $x = A\cos(\omega t+\alpha)$.

when $t=0$ $\quad x=0$

that is $A\cos\alpha = 0$ so $\alpha = \frac{\pi}{2} + k\pi$. $\quad k = 0, \pm1, \pm2, \cdots$

$A = 5cm$.

$T = \frac{1}{1} = 1s = \frac{2\pi}{\omega}$ so $\omega = 2\pi$.

so $x = 5\cos(2\pi t + \frac{\pi}{2} + k\pi)$

b. $t = \frac{8}{3}$

$x = 5\cos(2\pi \times \frac{8}{3} + \frac{\pi}{2} + k\pi) = \frac{5\sqrt{3}}{2}\cos k\pi$

$\frac{dx}{dt} = -5\times 2\pi \sin(2\pi t + \frac{\pi}{2} + k\pi)$

so when $t = \frac{8}{3}$

$\frac{dx}{dt} = -10\pi \sin(\frac{16}{3}\pi + \frac{\pi}{2} + k\pi) = 5\pi\cos k\pi$

$\frac{d^2x}{dt^2} = -20\pi^2\cos(2\pi t + \frac{\pi}{2} + k\pi)$

so when $t = \frac{8}{3}$

$\frac{d^2x}{dt^2} = -10\pi^2\sqrt{3}\cos k\pi = -10\sqrt{3}\pi^2\cos k\pi$

9. a. $E = \int k\left(\frac{1}{r_1} + \frac{1}{r_2}\right)^2 dA$.

where r_1 and r_2 has the dimension of L.

so $\left(\frac{1}{r_1} + \frac{1}{r_2}\right)^2$ has the dimension of L^{-2}.

and A has dimension of L^2

so $\left(\frac{1}{r_1} + \frac{1}{r_2}\right)^2 dA$ is dimensionless.

that is k has the same dimensions as Energy.

b. $F = \alpha r v \eta$ where α is a dimensionless constant.

dimensions of each parameter are.

F: MLT^{-2}

r: L

v: LT^{-1}

So the dimension of η is

$MLT^{-2} \cdot L^{-1} \cdot L^{-1}T = ML^{-1}T^{-1}$

that is η has dimensions of Mass / [Time x Length]

(c) We know from the previous part that $[\eta] = \frac{[M]}{[TL]}$ and $[\kappa] = [E] = \frac{[M][L]^2}{[T]^2}$. We want to solve for the exponents in the following equation,

$$[\tau] = [T] = [\lambda]^a[\kappa]^b[\eta]^c = [L]^a\left(\frac{[M][L]^2}{[T]^2}\right)^b\left(\frac{[M]}{[TL]}\right)^c.$$

This gives us three equations to solve corresponding to each unit

$$\begin{aligned} 0 &= b + c \\ 0 &= a + 2b - c \\ 1 &= -2b - c. \end{aligned}$$

Solving these equations tells us $b = -c$, $a = 3c$, and $c = 1$. So,

$$\tau \propto (\text{dimensionlessconstant})\frac{\eta\lambda^3}{\kappa}.$$

(d) If we added z to our list of parameters we would have another exponent, d, to solve for. However, z has the same units as λ so we still only have three fundamental units to make equations from. This gives us three equations and four unknowns,

$$\begin{aligned} 0 &= b + c \\ 0 &= a + 2b - c + d \\ 1 &= -2b - c. \end{aligned}$$

We cannot solve for all of the unknowns. The closest we can get is

$$\tau \propto (\text{dimensionlessconstant})\frac{\eta}{\kappa}\lambda^{\mathrm{a}}\mathrm{z}^{3-\mathrm{a}}.$$

Side note: A full hydrodynamic theory treatment of this problem reveals $a = 6$ and $d = -3$.