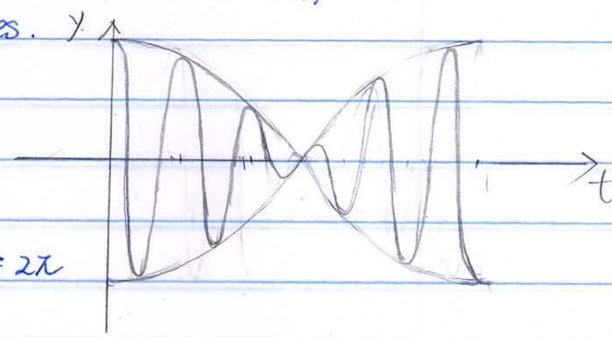
Physics 351 Problem Set 2 Solutions

1 French prob. 2-3.

$$\begin{cases} y_1 = A\cos 10\pi t \\ y_2 = A\cos 12\pi t \end{cases}$$

as we have known, the beat frequency is simply the difference of their individual frequencies.



so, we have

$$\omega_1 = 10\pi$$

$$\omega_2 = 12\pi$$

so $\omega_{beat} = |\omega_1 - \omega_2| = 2\pi$

$\therefore T = \frac{2\pi}{\omega} = \frac{2\pi}{2\pi} = 1\,s$

$y_1 + y_2 = (2A\cos\pi t)\cos 11\pi t$

2. A vertical mass & spring setup.

we can choose the y direction downwards as shown in the left graph, we assume the equilibrium is at $y = y_0$

then according to Newton's 2nd Law, we have

$$F = m\ddot{y} = mg - ky$$

at the static equilibrium point

$$m\ddot{y} = mg - ky_0 = 0$$

$$\therefore y_0 = \frac{mg}{k}$$

we can set $y' = y - y_0$

and insert this into the equation, then we can get

$$m\ddot{y}' = m(\ddot{y} - \ddot{y}_0) = m\ddot{y}$$

$$= mg - ky = mg - k(y' + y_0)$$

$$= -ky'$$

that is $m\ddot{y}' = -ky'$

That's the same equation just like the horizontal case.

So it has solution of

$y'(t) = A\cos(\omega t + \varphi)$

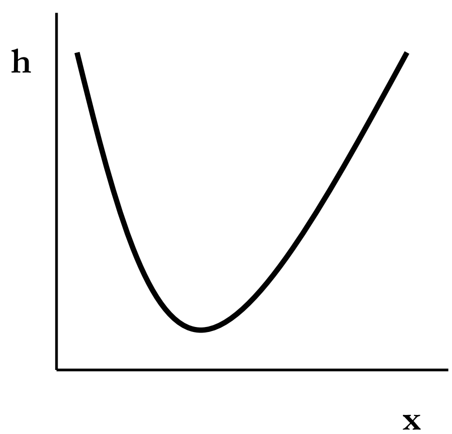
where $\omega = \sqrt{\frac{k}{m}}$

$y(t) = y'(t) + x_0$

$= A\cos(\omega t + \varphi) + \frac{mg}{k}$.

(3) A rolling ball.

(a) As x $\rightarrow 0$, h $\rightarrow +\infty$, and as x $\rightarrow +\infty$, h $\rightarrow +\infty$. Also there is only one value of x for which $\frac{dh}{dx} = -\frac{1}{2}\frac{a}{x^{3/2}} + b$ is zero, namely $x_0 = \left(\frac{a}{2b}\right)^{2/3}$. (From the first observation, or by looking at the second derivative of h, it should be clear that h(x_0) is a local minimum.) These two facts together mean h(x) must look like



You could also have deduced this just from the first statement about the limits of h(x) and the shape of the functions $\sqrt{x}$ and x.

(b) An equilibrium point is one at which the net force acting on an object is zero. Since $F = -\frac{dU}{dx}$, this is equivalent to the statement that an equilibrium point is one at which the spatial derivative of the potential energy function is zero. For our system, the potential energy is simply the gravitational potential energy ***U*(x) = m g h(x)**. Therefore, from (a), **there is only one equilibrium point.**

(c) From above, there is only one equilibrium point, $x_0 = \left(\frac{a}{2b}\right)^{2/3}$. The ball rolls back and forth about this point. How do we determine the frequency of oscillation? There are three reasons to think that a **Taylor series expansion** might help:

(i) We're asked to recall the lectures of Week 1, in which we discussed why "any" oscillation is harmonic.

(ii) We're asked to consider "small oscillations" about equilibrium, which begs for a Taylor series expansion. (Why?)
(iii) Most importantly: Our ball is oscillating in an "unfamiliar" potential energy landscape: $U(x) = mg\left[\frac{a}{\sqrt{x}} + bx\right]$. We could determine a differential equation describing its motion, but I wouldn't have the slightest idea what its solution would be. If only we could turn this *U*(x) into a form for which we know the solution, perhaps a form that looks like the oscillations of a mass on a spring. What is *U*(x) for a mass on a spring? Remembering it, or *very simply deriving it* from F = -kx, we know that $U_{spring} = (1/2)\ k\ x^2$, where x is the displacement from equilibrium. This is a simple quadratic polynomial. A Taylor series is a polynomial expansion. Eureka!

And so we expand. In general, as I sincerely hope you all know by now, the expansion of U(x) **about $x=x_0$** is

$$U(x) = U(x_0) + \left.\frac{dU}{dx}\right|_{x=x_0}(x-x_0) + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x=x_0}(x-x_0)^2 + \ldots$$

(higher order terms). Applying this to our *U*(x) expanded about our equilibrium point, we note that the $(x-x_0)$ term is zero, since the derivative of *U* is zero there, and so

$$U(x) = U(x_0) + 0 + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x=x_0}(x-x_0)^2 + \ldots$$

For small oscillations, we can neglect the higher order terms. So this looks just like $U_{spring} = (1/2)\ k\ x^2$, since it's just a constant times a quadratic displacement! You may be worried that there is a constant term $U(x_0)$ in the above expression that does not show up in U_{spring}, but this doesn't matter – one can arbitrarily add or subtract a constant from any potential energy function; since $F = -\frac{dU}{dx}$, these constants don't translate into a physically meaningful force. You may be worried that $U_{spring} = (1/2)\ k\ x^2$, while our expression involves $(x-x_0)^2$. But think about what these symbols *mean* – for the spring, x is the displacement from equilibrium, while for our ball, $(x-x_0)$ is the displacement from equilibrium. Hence our $(x-x_0)$ and the spring's x map onto the same physical concept.

Therefore our rolling ball "looks just like" a mass-on-a-spring, with an effective spring constant $k = \left.\frac{d^2U}{dx^2}\right|_{x=x_0}$. Explicitly, $k = \frac{3}{4}\frac{a}{x_0^{5/2}}mg = \frac{3}{4}\frac{(2b)^{5/3}}{a^{2/3}}mg$. Therefore the angular frequency ω=√("k"/m) and period T = 2π/ω are given by

$$T = 2\pi\sqrt{\frac{4a^{2/3}}{3g(2b)^{5/3}}} = \frac{2^{1/6}\pi a^{1/3}}{b^{5/6}}\sqrt{\frac{1}{3g}}.$$

(The last two expressions are equivalent; either is fine.)

Let's check the dimensions: h and x have dimensions of length, so, from the statement of h(x) it must be that $[a] = L^{3/2}$ and $[b] = 1$. Therefore $[period] = L^{1/2}\sqrt{\frac{1}{L/T^2}} = T$, as it should be.

Another approach:

(c). for small oscillations, we can make Taylor expansion around the equilibrium point for potential energy V.

$$V = mg\,h(x).$$

$$= mg\left(\frac{a}{\sqrt{x}} + bx\right)$$

$$= V(x_0) + \frac{1}{2} mg\, h''(x_0)(x-x_0)^2.$$

$$\therefore\ F = -\frac{\partial V(x)}{\partial x}$$

$$= -mg\, h''(x_0)(x-x_0)$$

$$\therefore\ m\ddot{x} = -mg\, h''(x_0)(x-x_0)$$

that is $\ddot{x} + g\,h''(x_0)\,x = g\,h''(x_0)\,x_0$

so $\omega = \sqrt{g\,h''(x_0)}$

$$\therefore\ T = \frac{2\pi}{\omega}$$

$$= \frac{2\pi}{\sqrt{g\,h''(x_0)}}$$

$$h''(x_0) = \frac{3}{4}a\left(\frac{a}{2b}\right)^{\frac{2}{3}\times(-\frac{5}{2})} = \frac{3}{4}a\left(\frac{a}{2b}\right)^{-\frac{5}{3}} = \frac{3}{4}a^{-\frac{2}{3}}(2b)^{\frac{5}{3}}$$

$$= \frac{3}{2}\left(\frac{2b}{a}\right)^{\frac{2}{3}} b$$

$$\therefore\ T = 2\pi\left(g\cdot\frac{3}{2}\left(\frac{2b}{a}\right)^{\frac{2}{3}} b\right)^{-\frac{1}{2}}$$

$$= 2^{\frac{1}{6}}\cdot\frac{2\pi}{\sqrt{3g}}\, a^{\frac{1}{3}} b^{-\frac{5}{6}}.$$

4 for back-and-forth swaying, we have

$$T_1 = 2\pi\sqrt{\frac{L_0}{g}}.$$

for up-down oscillation, we have

$$T_2 = 2\pi\sqrt{\frac{mL_0}{AY}}.$$

and $T_1 = 200\,T_2$.

$$\therefore \sqrt{\frac{L_0}{g}} = 200\sqrt{\frac{mL_0}{AY}}$$

$$\therefore Y = \frac{mg}{A} \times 4\times10^4$$

$$A = \pi r^2 = \pi\left(\frac{d}{2}\right)^2 = 3.14 \times \left(\frac{0.1\times10^{-3}}{2}\right)^2 = 7.85\times10^{-9}\ m^2$$

and we can guess that a spider has a volume of about $1\,cm^3$, and has the same density as water,

so $m \approx 1\,g/cm^3 \times 1\,cm^3 = 1g = 10^{-3}\,kg$.

$$\therefore Y = \frac{10^{-3}\,kg \times 10\ m\cdot s^{-2}}{7.85\times10^{-9}\ m^2} \times 4\times10^4$$

$$= 1.27\times10^6\ kg\cdot m^{-1}\cdot s^{-2} \times 4\times10^4$$

$$= 5.08\times10^{10}\ kg\cdot m^{-1}\cdot s^{-2}$$

$$\doteq 5.1\times10^{10}\ \frac{N}{m^2}$$

5. French problem 3-3a.

we set y direction upwards as shown in the left graph.

then we have

$$\omega = 2\pi f = 2\pi \cdot \frac{10}{\pi} = 20.$$

$\therefore\ y(t) = 5\cos(20t+\phi)$

at $t=0$

$$y(0) = 5\cos\phi = -5.$$

$$\cos\phi = -1$$

so we can choose $\phi = \pi$.

$\therefore\ y(t) = 5\cos(20t+\pi)$

at first, the velocities of the platform and of block are the same, when the system moves above the equilibrium position, the acceleration will be in the downwards direction,

so when the acceleration of the platform exceeds that of the block, the block will leave the platform.

$a_{block} = -g$.

$$a_{platform} = \ddot{y}(t) = -20^2 \times 5\text{cm}\cos(20t+\phi)$$
$$= -20\cos(20t+\phi)$$

$a_{platform} = a_{block}$

$\therefore\ 20\cos(20t+\phi) = 10 \qquad \therefore \cos(20t+\phi) = \frac{1}{2}$

$$\therefore\ y = 5\cos(20t+\phi)$$
$$= 5\times\frac{1}{2} = 2.5\text{ cm}.$$