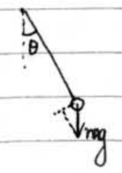
Prof. Raghuveer Parthasarathy

Problem Set 3 -- Solutions -- Physics 351 Fall 2007

1\. A pendulum



a. according to Newton's 2nd Law

$F = mg\sin\theta = ma = -mL\ddot{\theta}$

$\therefore \ddot{\theta} = -\frac{g}{L}\sin\theta \doteq -\frac{g}{L}\theta$

$\theta = \theta_0 \cos\left(\sqrt{\frac{g}{L}}\, t + \varphi\right)$

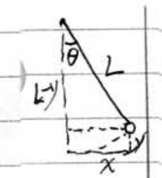
when $t=0$ $\quad \dot{\theta}(0) = \theta_0\left(-\sqrt{\frac{g}{L}}\right)\sin\left(\sqrt{\frac{g}{L}}\, t + \varphi\right)\Big|_{t=0} = 0$

$\therefore \sin\varphi = 0$

we can choose $\varphi = 0$

$\therefore \theta = \theta_0 \cos\left(\sqrt{\frac{g}{L}}\, t\right)$

b. we have the energy conservation



$mgy + \frac{1}{2}mv^2 = E$

here we assume $y \ll x$.

then we have

$y = L(1-\cos\theta) = L\cdot 2\sin^2\frac{\theta}{2}$

$\approx \frac{1}{2}L\theta^2$

$v \approx L\dot{\theta}$

$\therefore \frac{1}{2}mL^2\dot{\theta}^2 + \frac{1}{2}mgL\theta^2 = E$

$\therefore \theta = A\cos(wt+\varphi)$, where $w = \sqrt{\frac{g}{L}}$

when $t=0$ $\quad \dot{\theta}(t=0) = -wA\sin\varphi = 0$

$\theta(t=0) = A\cos\varphi = \theta_0$

$\therefore \varphi = 0 \quad A = \theta_0$

$\therefore \theta = \theta_0\cos(wt)$, $w = \sqrt{\frac{g}{L}}$

(c) by energy conservation

$mgL(1-\cos\theta) = \frac{1}{2}mv^2$

$\therefore v = \sqrt{2gL(1-\cos\theta)} = \sqrt{2\times 10\times 1(1-\cos 15^\circ)} = \sqrt{20\times\left(1-\sqrt{\frac{1+\sqrt{3}/2}{2}}\right)} \doteq 0.83 (m/s)$

2

(a) The volume of water must be conserved.

volume $\pi r^2(-y_1)$ "leaves" the left,

and $\pi(\alpha r)^2 y_2$ "enters" the right,

So $\pi r^2(-y_1) = \pi\alpha^2 r^2 y_2 \Rightarrow \underline{y_2 = -y_1/\alpha^2}$

(b) gravitational potential energy "U = "mgh".

At equilibrium $(y_1 = y_2 = 0)$, define $U \equiv 0$.

Let's calculate the gain in potential energy associated with moving a disk of thickness dy_1 at height y_1 to height $y_2 = -y_1/\alpha^2$. (from part (a), this will conserve volume.)

$$dU = \text{"mgh"} = \rho \underbrace{\pi r^2}_{\text{"A"}} dy_1\, g(y_2 - y_1)$$

$$= -\rho\pi r^2 g y_1\left(\frac{1}{\alpha^2}+1\right)dy_1 \quad \text{(note } y_1 < 0 \Leftrightarrow dU_1 > 0\text{)}.$$

The total pot'l energy $U = \int dU$

$$= \int_{-y_1}^{0} -g\rho\pi r^2 y_1\left(\frac{1}{\alpha^2}+1\right)dy_1 = \underline{g\rho\pi r^2\left(1+\frac{1}{\alpha^2}\right)y_1^2 = U}$$

some negative value

(c) Kinetic energy = "$\frac{1}{2}mv^2$"

Left arm: $KE = \frac{1}{2}\underbrace{\rho\pi r^2(h+y_1)}_{\text{"m"}}\dot{y}_1^2$

middle: $KE = \frac{1}{2}\rho\pi r^2 d\,\dot{y}_1^2$

same velocity as left arm, since same radius

Right arm: $KE = \frac{1}{2}\rho\pi(\alpha r)^2(h+y_2)\dot{y}_2^2$

$= KE = \frac{1}{2}\rho\pi\alpha^2 r^2\left(h-\frac{y_1}{\alpha^2}\right)\frac{1}{\alpha^4}\dot{y}_1^2$

Total: $KE = \frac{1}{2}\rho\pi r^2\left[h+y_1+d+\frac{1}{\alpha^2}\left(h-\frac{y_1}{\alpha^2}\right)\right]\dot{y}_1^2$

(d) Total Energy $E = U + KE$ = constant

$E = g\rho\pi r^2\left(1+\frac{1}{\alpha^2}\right)y_1^2 + \frac{1}{2}\rho\pi r^2\left[h+y_1+d+\frac{1}{\alpha^2}\left(h-\frac{y_1}{\alpha^2}\right)\right]\dot{y}_1^2$

ugly!

Assume $y_1 \ll h$ & $y_2 \ll h$ (as instructed).

$\Rightarrow E = g\rho\pi r^2\left(1+\frac{1}{\alpha^2}\right)y_1^2 + \frac{1}{2}\rho\pi r^2\left[h\left(1+\frac{y_1}{h}\right)+d + \frac{1}{\alpha^2}h\left(1-\frac{y_1}{h\alpha^2}\right)\right]\dot{y}_1^2$

$1+\frac{y_1}{h} \approx 1$

$\Rightarrow E \approx g\rho\pi r^2\left(1+\frac{1}{\alpha^2}\right)y_1^2 + \frac{1}{2}\rho\pi r^2\left[h+d+\frac{h}{\alpha^2}\right]\dot{y}_1^2$

This looks just like a simple harmonic oscillator: $E = \frac{1}{2}ky^2 + \frac{1}{2}m\dot{y}^2$!

"k" $= g2\rho\pi r^2\left(1+\frac{1}{\alpha^2}\right)$, "$m$" $= \rho\pi r^2\left(d+\left(1+\frac{1}{\alpha^2}\right)h\right)$

Therefore the angular frequency $\omega = \sqrt{\text{"}k\text{"}/\text{"}m\text{"}}$

$\rightarrow$

$$\rightarrow \omega = \sqrt{\frac{2(1+1/\alpha^2)g}{d+h(1+1/\alpha^2)}}$$

If $\alpha = 1$, $\omega \rightarrow \sqrt{\frac{2(1+1)g}{d+h(1+1)}} = \sqrt{\frac{2g}{d+2h}}$

$= \sqrt{2g/L}$, where L is the length of the "simple" u-tube.

3 Average Energy:

(a) The quick method:

A harmonic oscillator is described by $x(t) = A\cos(\omega t + \phi)$. Its potential energy

$$P(t) = (1/2)kx^2 = (1/2)kA^2\cos^2(\omega t + \phi), \tag{1}$$

and its kinetic energy

$$\begin{aligned} K(t) &= (1/2)m\dot{x}^2 \\ &= (1/2)mA^2\omega^2\sin^2(\omega t + \phi) \\ &= (1/2)kA^2\sin^2(\omega t + \phi), \end{aligned}$$

where we note that $\omega^2 = k/m$. **2 points up to here.**

Sine and cosine functions have the same shape, just offset by $\pi/2$ in phase. Therefore, averaged over a complete cycle, $\sin^2()$ and $\cos^2()$ have the same mean value. Moreover, we recall from elementary trigonometry that $\sin^2(y) + \cos^2(y) = 1$ (for whatever y), so $<\sin^2(y)> + <\cos^2(y)> = <1>$, where $<>$ means "average," so $<\sin^2(y)> = <\cos^2(y)> = 1/2$. Therefore

$$<P> = <K> = (1/2)kA^2(1/2) = (1/4)kA^2.$$

(b) The verbose method:

3. Average Energy.

$$x(t) = A\sin(\omega t - \varphi)$$

$$\therefore \dot{x}(t) = \omega A\cos(\omega t - \varphi)$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2(t) = \frac{1}{2}m\omega^2A^2\cos^2(\omega t - \varphi)$$

$$V = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\sin^2(\omega t - \varphi)$$

$$\bar{V} = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} V\,dt = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{1}{2}kA^2\sin^2(\omega t - \varphi)\,dt$$

$$= \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{1}{2}kA^2\frac{1}{2}(1-\cos 2(\omega t - \varphi))\,dt$$

$$= \frac{\omega}{2\pi}\cdot\frac{1}{4}kA^2\cdot\frac{2\pi}{\omega}$$

$$= \frac{1}{4}kA^2.$$

$$\bar{K} = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} K\,dt$$

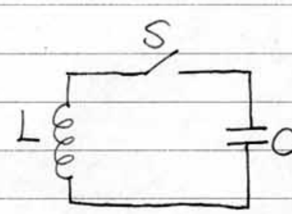
$$= \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{1}{2}m\omega^2A^2\cos^2(\omega t - \varphi)$$

$$= \frac{1}{4}m\omega^2A^2$$

$$\because \omega = \sqrt{\frac{k}{m}} \quad \therefore m\omega^2 = k$$

$$\therefore \bar{K} = \frac{1}{4}kA^2.$$

4 a. we have



$$V_L = L\frac{dI}{dt}, \quad V_C = \frac{Q}{C}$$

$$I = \frac{dQ}{dt}.$$

$$\Sigma V = V_L + V_C = L\frac{dI}{dt} + \frac{Q}{C}$$

$$= L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

$$\therefore \ddot{Q} + \frac{1}{LC}Q = 0.$$

it have the same form as the mass on a spring equation.

b $\ddot{Q} + \frac{1}{LC} Q = 0$

$\ddot{Q} + \omega^2 Q = 0$

$\omega = \frac{1}{\sqrt{LC}}$

c. $Q(t) = A\cos\left(\frac{1}{\sqrt{LC}} t + \varphi\right)$

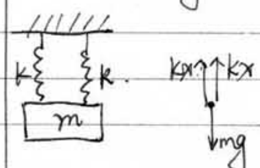
at $t = 0$. $V_c = V_0$

$\therefore Q = CV_c = CV_0$

$Q(t=0) = Q_{max}$

$\therefore Q(t) = CV_0 \cos\left(\frac{1}{\sqrt{LC}} t\right)$

5. Arrangments of springs.

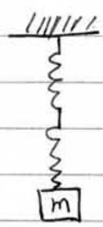


by Newton's 2nd Law, we have

$ma = -kx - kx - mg.$

$ma + 2kx = -mg$

$\therefore \omega = \sqrt{\frac{2k}{m}}.$



if we want move the mass by ~~the~~ same amount with that of single spring case,

$-mg - kx = ma$

$\therefore x = \frac{-mg - ma}{k}$

assume K is the combined spring constant,

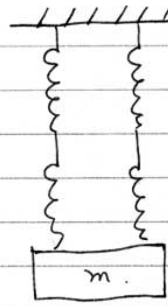
$\therefore$ we have

$$\frac{-mg - ma}{K} = \frac{-mg - ma}{k} + \frac{-mg - ma}{k}$$

$\therefore K = \frac{k}{2}$

$\therefore \omega = \sqrt{\frac{k}{2m}}.$

c.



6. Critical Damping.

$$x(t) = (A + Bt)\, e^{-\gamma t/2}.$$

$$\gamma = 2\omega_0.$$

$$\frac{dx}{dt} = B e^{-\gamma t/2} + (A+Bt)\left(-\frac{\gamma}{2}\right) e^{-\frac{\gamma t}{2}}$$

$$\frac{d^2x}{dt^2} = -\frac{\gamma}{2} B e^{-\frac{\gamma t}{2}} + -\frac{\gamma}{2} B\, e^{-\frac{\gamma t}{2}} + (A+Bt)\,\frac{\gamma^2}{4} e^{-\frac{\gamma t}{2}}$$

$$\therefore \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x$$

$$= -\gamma B e^{-\frac{\gamma t}{2}} + (A+Bt)\frac{\gamma^2}{4} e^{-\frac{\gamma t}{2}} + \gamma B e^{-\frac{\gamma t}{2}} + (A+Bt)\left(-\frac{\gamma^2}{2}\right) e^{-\frac{\gamma t}{2}} + \frac{\gamma^2}{4}(A+Bt)e^{-\frac{\gamma t}{2}}$$

$$= 0.$$

so. $x(t) = (A+Bt)\, e^{-\frac{\gamma t}{2}}$ is a solution to the equation of motion $\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$.