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University of Oregon; Fall 2007

Physics 351 – Vibrations and Waves

Problem Set 4

Due date: Friday, Oct. 26, 5 pm.

Reading: French Chapter 3; start Chapter 4.

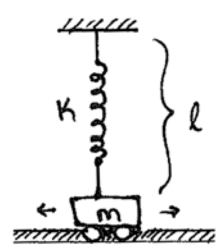
(1, 3 pts.**) A Tuning Fork.** A tuning fork rings at "A above middle C," a frequency of $f = 440$ Hz. (Note that this is the frequency, f, not the angular frequency, ω.) The intensity of the sound, which is proportional to the energy of the oscillating fork, decreases by factor of 5 in 4 seconds.

- **(a,** 1 pt.**)** Without doing any math, except perhaps some simple arithmetic, we know that the system is weakly damped. Why?
- **(b,** 2 pts.**)** What is the Q of the tuning fork?

(2, 5 pts.**)** An object of mass 0.3 kg is attached to a spring of spring constant 240 N/m. The object is subject to a resistive force given by $F = -bv$, where v is the velocity in meters per second.

- **(**a, 2 pts.**)** If the damped frequency is $\sqrt{7}/3$ times the frequency in the absence of damping, what is the value of the damping constant, b?
- **(**b, 2 pts.**)** What is the Q of the system?
- **(**c, 1 pt.**)** By what factor is the amplitude of the oscillation reduced after 10 cycles?

(3, 6 pts.**) A tethered cart on a track.** A cart of mass m can roll along a frictionless one-dimensional track, but cannot leave the track. Its top is tethered to a spring of spring constant k that is attached to a fixed point. The spring is stretched to a length l that is much larger than the equilibrium (unstretched) spring length l_0. See figure. Think about how the "smallness" of the oscillations simplifies the trigonometry of the setup. (Hint: only the lowest order terms in expansions as a function of (x/l) are relevant.)



- **(a,** 2 pts.**)** Write down the differential equation of motion (i.e. apply Newton's Laws) for small oscillations of the cart about x=0, the equilibrium position. (There is no damping.)
- **(b,** 2 pts.**)** Determine the angular frequency of small oscillations, ω.
- **(c,** 2 pts.**)** Now let's introduce damping: Due to friction, the cart's motion is damped by a drag force $F = -bv$, where v is the velocity in the x-direction. If l is increased, will the Q-factor of the oscillations increase or decrease?

(4, 6 pts.**) Initial conditions.** Consider our "usual" damped mass on a spring, i.e. the damping force is $F = -bv$ and the restoring force is $F = -kx$, where x is position and v is velocity. As usual, you can define $\omega_0^2 = \frac{k}{m}$ and $\gamma = \frac{b}{m}$. At time t=0 the mass is released from position $x = x_0$ with zero velocity.

- **(a,** 2 pts.**)** Suppose the damping is such that the system is **underdamped**. We know, from class, that the general solution to the equation of motion can be written $x(t) = Ae^{-\gamma t/2}\cos(\omega t - \varphi)$. Find the *particular* solution x(t) for these initial conditions. In other words, find the particular amplitude factor "A" and phase offset "φ." *Hints*: (*i*) Don't be dismayed if your answer isn't as "clean" as that of an undamped oscillator. (*ii*) Recall trigonometry. If tan(α)=a/b, what is sin(α)? (I can never remember, which is why I draw triangles and figure out the answer.) (*iii*) Two ways to check your answer: You should be able to show that if $\gamma \to 0$, you recover the simple, expected x(t). And if γ=2ω, your phase offset should be π/4 radians and A should be $x_0\sqrt{2}$.
- **(b,** 2 pts.**)** Suppose b is such that the system is **overdamped**. Find the *particular* solution x(t) for these initial conditions. (i.e. find the two undetermined parameters to the general x(t) equation.)
- **(c,** 2 pts.**)** Suppose b is such that the system is **critically damped**. Find the *particular* solution x(t) for these initial conditions.

(5, 10 pts.**) Critical damping.** I claimed in class that the critically damped oscillator exhibits the fastest decay to equilibrium without overshooting – in other words, it is faster than the overdamped oscillator. This may seem counter-intuitive – shouldn't the overdamped oscillator be "squashed" quickly to x=0? The issue is important, since in all sorts of oscillators ranging from shock absorbers in cars to electrical circuits, one often wants deviations from equilibrium to die out as quickly as possible. Consider the setup of Problem 4, in which x(t=0) = x_0 and v(t=0)=0. **Prove that the critically damped oscillator decays more rapidly than the overdamped oscillator.** There are several ways to do this; I suggest proving that for all times, t, x(t) for the overdamped system is greater than x(t) for the critically damped system. This is a challenging problem; if you find that this is the first problem set on which you can't answer all the problems, don't be too sad. One hint: You may wish to expand in a Taylor series in βt, where $\beta = \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$, though we're *not* limiting ourselves to small βt so you can't neglect any terms in the infinite series expansion.

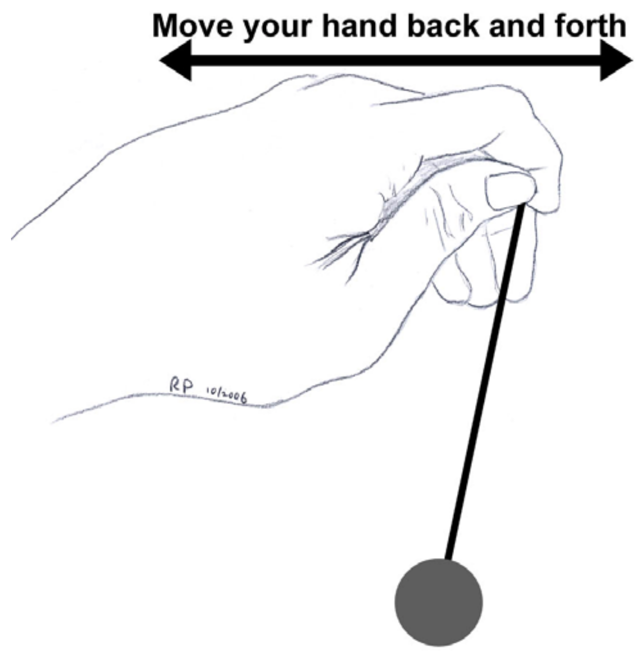
(6, 7 pts.**) Radiation from an accelerated charge.** Mr. K. builds an oscillator consisting of an electron oscillating back and forth harmonically with frequency f and amplitude A, and with clever combinations of vacuums and electric fields manages to eliminate all friction from the system. (Note that f is the frequency, not the angular frequency.) "At last," he says, "a perfectly undamped oscillator!" He doesn't realize, however, that any electrically charged object, when accelerated, will radiate energy in the form of electromagnetic waves. The radiated power is $P = \frac{2}{3}\frac{1}{4\pi\varepsilon_0}\frac{q^2 a^2}{c^3}$, where q is the charge, a is the acceleration, and c is the speed of light. An electron has charge q = 1.60 x 10^{-19} C and mass m = 9.11 x 10^{-31} kg.

(a, 3 pts.**)** What fraction of the initial energy of the oscillator is radiated away after one cycle? (You may find the math of PS3 useful.) Is the damping weak or strong?

(b, 3 pts.**)** Estimate the Q of this oscillator in terms of q, f, etc. *Hint*: Recall the Taylor expansion of e^y; for small y, what is the lowest order term in y? You'll use m in your expression for the initial energy of the electron.

(c, 1 pt.**)** Evaluate your answer from (c) numerically for f corresponding to the oscillation frequency of red light. (Red light has a wavelength around $\lambda \approx 650 \times 10^{-9}$ m; wavelength and frequency are related by $\lambda f = c$.) Think about whether the Q value you find is consistent with your conclusions for part (a).

(7, 9pts: 3,6**) Resonance of a simple pendulum – experiment.** In class, we've begun to discuss the effect of periodic driving forces on oscillating systems. The most important concepts that emerge are those of "resonance" and differences in response above and below resonance frequencies. Here, we'll try to develop some intuition about resonance. For this exercise you'll do a simple experiment and describe the results. Write at most 300 words describing what you've done and what you've observed. You'll be graded on the content of your response as well as the clarity of your writing. Type your answer. Communication skills, by the way, are vital for any practicing scientist – the majority of your professor's time, for example, is spent reading and writing scientific papers, preparing presentations, and writing long grant applications.



To start, create a simple pendulum – a mass on a string – for example, keys at the end of a shoelace. As you're holding the end of the string, move your hand back and forth (horizontally), varying the period of your hand's oscillations (see the figure). Watch the response of the mass. Try frequencies that are small and large. Describe the amplitude of the motion of the mass (i.e. how far it's moving back and forth) and the "phase" relative to your hand (i.e. if it moves in the same direction as your hand or opposite to it). By watching the phase response as a function of your hand's driving frequency, estimate the natural frequency of the simple pendulum. (Admittedly, this is not a very precise way to measure the natural frequency!) Compare your estimate with what you'd "expect" for the pendulum (i.e. the theoretical value of the natural frequency).