Prof. Raghuveer Parthasarathy
University of Oregon; Fall 2007
Physics 351 – Vibrations and Waves

Problem Set 4: SOLUTIONS

(1) A Tuning Fork.

(a) The tuning fork is underdamped (or else we couldn't hear a continuous tone, because there wouldn't be any oscillations of the air). Since f = 440 Hz, the period T =1/f is around 2 ms. The tone persists for several seconds, so the decay time is much larger than the oscillation time. Therefore the system is weakly damped.

(b) The energy of an underdamped oscillator decays according to $E = E_{t=0}e^{-\gamma t}$. At time t=4s, the energy is 5 times smaller than its initial value, so $E_{t=4} = E_{t=0}/5$, so $e^{-\gamma 4} = 1/5$. Therefore $\gamma = \log(5)/4 \approx 0.4$ s^{-1}. The frequency f = 440 Hz, and by definition $\omega = 2\pi f$. The Q factor $Q = \omega_0/\gamma$. Since the system is weakly damped, we know that $\omega \approx \omega_0$. (If you don't believe this, use the above value of γ and solve $\omega = \sqrt{\omega_0^2 - \gamma^2/4}$ for ω_0.) Therefore $Q \approx \dfrac{2\pi 440}{\log(5)/4} \approx 7000.$

(2) [next page]

2. $m = 0.3\,kg$. $k = 240\,N/m$.

(a) $m\ddot{x} = -kx - bv$.

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} = \frac{k}{m} - \frac{b^2}{4m^2} = \left(\frac{\sqrt{7}}{3}\omega_0\right)^2 = \frac{7}{9}\frac{k}{m}$$

$$\therefore \frac{b^2}{4m^2} = \frac{2}{9}\frac{k}{m}$$

$$\therefore b = \left(\frac{8km}{9}\right)^{1/2} = \left(\frac{8\times 240\times 0.3}{9}\right)^{1/2} = 8$$

(b) $$\omega^2 = \omega_0^2\left(1 - \frac{1}{4Q^2}\right) = \frac{7}{9}\omega_0^2$$

$$\therefore 1 - \frac{1}{4Q^2} = \frac{7}{9}$$

$$\therefore Q^2 = \frac{1}{4}\times\frac{9}{2} = \frac{9}{8}$$

$$\therefore Q = \frac{3}{\sqrt{8}} \doteq 1.06$$

(c) $$\gamma = \frac{\omega_0}{Q} = \frac{\sqrt{240/0.3}}{\frac{3}{\sqrt{8}}} = \frac{80}{3}.$$

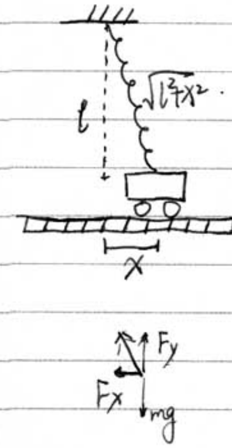
$$\therefore t = 10T = 10\times\frac{2\pi}{\omega} = 10\times\frac{2\pi}{\frac{\sqrt{7}}{3}\sqrt{\frac{240}{0.3}}} = \frac{6\pi}{\sqrt{56}}$$

$$\therefore \frac{A(t)}{A_0} = e^{-\gamma t/2} = e^{-\frac{80}{3}\times\frac{6\pi}{\sqrt{56}}\times\frac{1}{2}}$$

$$\doteq e^{-33.59}$$

$$\doteq 2.6\times 10^{-15}$$

(3)



3. $F_x = -k\left(\sqrt{x^2+l^2} - l_0\right) \cdot \frac{x}{\sqrt{x^2+l^2}}$

$= -kx + kl_0 x \left(l^2 (1+(\frac{x}{l})^2) \right)^{-\frac{1}{2}}$

$\doteq -kx + k\frac{l_0}{l} x$

$= -kx \frac{l-l_0}{l}$

so the differential equation of motion is.

$m\ddot{x} + kx\frac{l-l_0}{l} = 0.$

(b) the angular frequency of small oscillations

$\omega = \sqrt{\frac{l-l_0}{l} \cdot \frac{k}{m}}$

(c) if we have drag force $F = -bv$ due to friction,
the differential equation of motion is

$m\ddot{x} + b\dot{x} + \frac{k(l-l_0)}{l} x = 0.$

$\gamma = \frac{b}{m}$ is constant.

$Q = \frac{\omega_0}{\gamma}$

$\omega_0 = \sqrt{\left(1-\frac{l_0}{l}\right)\frac{k}{m}}$

so if l is increased, ω_0 will also be increased.

$\therefore$ the Q-factor of the oscillations also increases.

(4) @ $t=0$, $x=x_0$, $\dot{x}=0$.

(a) underdamped oscillator $(\omega_0^2 > \gamma^2/4)$

$\Rightarrow$ general solution $x(t) = A e^{-\gamma t/2} \cos(\omega t - \phi)$,

where $\omega = \sqrt{\omega_0^2 - \gamma^2/4}$, $\omega_0^2 = k/m$, $\gamma = b/m$.

$x(t=0) = A\cos(-\phi) = x_0$ [Equation 1]

$\dot{x}(t) = A\left(-\frac{\gamma}{2}\right)e^{-\gamma t/2}\cos(\omega t - \phi)$

$+ A e^{-\gamma t/2}\omega(-1)\sin(\omega t - \phi)$

$\dot{x}(t=0) = -\frac{A\gamma}{2}\cos(-\phi) - A\omega\sin(-\phi)$

$\dot{x}(t=0) = 0$, and $\left.\begin{matrix}\cos(-\phi) = \cos(\phi) \\ \sin(-\phi) = -\sin(\phi)\end{matrix}\right\}$ even & odd properties

$\Rightarrow \frac{A\gamma}{2}\cos(\phi) = A\omega\sin(\phi)$ [Equation 2]

$\Rightarrow \tan(\phi) = \frac{\gamma}{2\omega}$

Returning to Equation 1, $A\cos(\phi) = x_0$

If $\tan(\phi) = \frac{\gamma}{2\omega}$, 

, then $\cos(\phi) = \frac{2\omega}{\sqrt{\gamma^2 + 2\omega^2}} = \frac{\omega}{\sqrt{\omega^2 + \gamma^2/4}}$

$$\Rightarrow \boxed{\begin{aligned} A &= x_0 \frac{\sqrt{\omega^2 + \gamma^2/4}}{\omega} \\ \phi &= \arctan\left(\frac{\gamma}{2\omega}\right) \end{aligned}} \Bigg\}$$

★ Gen. soln. $x(t) = Ae^{-\gamma t/2}\cos(\omega t - \phi)$ with these A, ϕ

Note that if $\gamma \to 0$, $\omega \to \omega_0$ and $\phi \to \arctan 0 = 0$,

and $A \to x_0$, so $x(t) = x_0\cos(\omega_0 t)$

as expected .

(b) overdamped oscillator. $\omega_0^2 < \gamma^2/4$.

$$\beta \equiv \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2}$$

General solution $x(t) = A_1 e^{-(\frac{\gamma}{2}+\beta)t} + A_2 e^{-(\frac{\gamma}{2}-\beta)t}$

$$\dot{x}(t) = -A_1\left(\frac{\gamma}{2}+\beta\right)e^{-(\frac{\gamma}{2}+\beta)t} - A_2\left(\frac{\gamma}{2}-\beta\right)e^{-(\frac{\gamma}{2}-\beta)t}$$

$$x(t=0) = x_0 \Rightarrow \quad A_1 + A_2 = x_0$$

$$\dot{x}(t=0) = 0 \Rightarrow \quad -A_1\left(\frac{\gamma}{2}+\beta\right) - A_2\left(\frac{\gamma}{2}-\beta\right) = 0$$

$$\Rightarrow (A_1 + A_2)\frac{\gamma}{2} + \beta(A_1 - A_2) = 0$$

$$\Rightarrow x_0\frac{\gamma}{2} + \beta(2A_1 - x_0) = 0$$

$$\Rightarrow 2\beta A_1 = \beta x_0 - \frac{\gamma}{2}x_0$$

$$\Rightarrow A_1 = x_0\left(\frac{1}{2} - \frac{\gamma}{4\beta}\right)$$

and $A_2 = x_0\left(\frac{1}{2} + \frac{\gamma}{4\beta}\right)$ ★

Gen. soln. $x(t) = A_1 e^{-(\frac{\gamma}{2}+\beta)t} + A_2 e^{-(\frac{\gamma}{2}-\beta)t}$

with these A_1, A_2.

Explicitly:

$$x(t) = x_0\left(\frac{1}{2} - \frac{\gamma}{4\beta}\right)e^{-(\frac{\gamma}{2}+\beta)t} + x_0\left(\frac{1}{2} + \frac{\gamma}{4\beta}\right)e^{-(\frac{\gamma}{2}-\beta)t}$$

$$= x_0 e^{-\frac{\gamma}{2}t}\left[\left(\frac{1}{2} - \frac{\gamma}{4\beta}\right)e^{-\beta t} + \left(\frac{1}{2} + \frac{\gamma}{4\beta}\right)e^{+\beta t}\right]$$

(c) critically damped oscillator. $\omega_0^2 = \gamma^2/4$.

$x(t) = (A + Bt)e^{-\gamma t/2}$ (gen. soln.).

$\dot{x}(t) = -\frac{\gamma}{2}(A+Bt)e^{-\gamma t/2} + Be^{-\gamma t/2}$

$x(t=0) = x_0 \Rightarrow A = x_0$. $\dot{x}(t=0) = 0 \Rightarrow -\frac{\gamma}{2}A + B = 0$

$\Rightarrow B = \frac{\gamma}{2}A$.

$$\Rightarrow \boxed{\begin{array}{l} x(t) = (A + Bt)e^{-\gamma t/2} \text{ with } A = x_0,\ B = \frac{\gamma}{2}x_0 . \\ x(t) = x_0\left(1 + \frac{\gamma t}{2}\right)e^{-\gamma t/2} \end{array}}$$

For this particular solution, $\dot{x}(t) = -\frac{\gamma}{2}(x_0 + \frac{\gamma}{2}t)e^{-\gamma t/2} + x_0\frac{\gamma}{2}e^{-\gamma t/2}$

$= \dot{x}(t) = -\frac{\gamma}{2}\, t e^{-\gamma t/2}$

we see that $\dot{x}(t)$ is always negative, and never zero.

(5) We'll make use of our solution to #4

Let's expand our overdamped $x(t)$ in powers of βt.
Why? We're comparing the overdamped ($\beta \neq 0$) to critically damped ($\beta = 0$) case, so we suspect that expanding in β may be useful.

Taylor expansion: (note $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$)

$$x(t) = X_0 e^{-\gamma t/2} \Big\{ \left(\tfrac{1}{2} - \tfrac{\gamma}{4\beta}\right)\left(1 - \beta t + \tfrac{1}{2}(\beta t)^2 - \tfrac{1}{6}(\beta t)^3 + \dots\right) + \left(\tfrac{1}{2} + \tfrac{\gamma}{4\beta}\right)\left(1 + \beta t + \tfrac{1}{2}(\beta t)^2 + \tfrac{1}{6}(\beta t)^3 + \dots\right)$$

$$= X_0 e^{-\gamma t/2} \Big\{ \left(\tfrac{1}{2}\right)\left(1 - \beta t + \tfrac{1}{2}(\beta t)^2 - \tfrac{1}{6}(\beta t)^3 + \dots + 1 + \beta t + \tfrac{1}{2}(\beta t)^2 + \tfrac{1}{6}(\beta t)^3 + \dots\right) + \left(\tfrac{\gamma}{4\beta}\right)\left(-1 + \beta t - \tfrac{1}{2}(\beta t)^2 + \tfrac{1}{6}(\beta t)^3 + \dots + 1 + \beta t + \tfrac{1}{2}(\beta t)^2 + \tfrac{1}{6}(\,)^3 + \dots\right) \Big\}$$

Note the cancellation of all the odd βt powers (first line) & even βt powers (second line)

$$\Rightarrow x(t) = X_0 e^{-\gamma t/2} \Big\{ 1 + \tfrac{1}{2}(\beta t)^2 + \tfrac{1}{24}(\beta t)^4 + \dots + \tfrac{\gamma}{2\beta}\left((\beta t) + \tfrac{1}{6}(\beta t)^3 + \tfrac{1}{120}(\beta t)^5 + \dots\right) \Big\}$$

$\beta > 0$ and $t > 0$, so all the terms in $\{\}$ are positive.

Therefore $\boxed{x(t) = X_0 e^{-\gamma t/2} \{\text{positive number}\} > X_0 e^{-\gamma t/2} \quad \left(\begin{smallmatrix}\text{for } t>0 \\ \beta>0\end{smallmatrix}\right)}$

Therefore $\boxed{x(t)_{\text{OVERDAMPED}} > x(t)_{\text{CRITICAL}}}$

(if $\beta = 0$, $x(t)_{\text{OVERDAMPED}} = x(t)_{\text{CRITICAL}}$)

(6) Radiation from an accelerated charge. Our electron is oscillating harmonically – therefore we already know that $x(t) = A\sin(2\pi ft)$, where I've set the phase offset to zero because it won't affect anything. (If this bothers you, go ahead and write it in.)

(a) We recall that power is the rate of change of energy. We are given the rate of energy loss as $\frac{dE}{dt} = \frac{2}{3}\frac{1}{4\pi\epsilon_0}\frac{q^2a^2}{c^3}$. The amount of energy lost over one cycle is $\Delta E = \int_0^T \frac{dE}{dt}dt$. To calculate this integral we need to know the acceleration of the particle, *a*., which follows simply from *x*(t): $\ddot{x} = (2\pi f)^2 A\sin(2\pi ft) = a$. Returning to the integral we have,

$$\Delta E = \int_0^T \frac{dE}{dt}dt = \int_0^T \frac{2}{3}\frac{1}{4\pi\epsilon_0}\frac{q^2}{c^3}(2\pi f)^4 A^2 \sin^2(2\pi ft)dt = \frac{2}{3}\frac{1}{4\pi\epsilon_0}\frac{q^2}{c^3}(2\pi f)^4 A^2 \int_0^T \sin^2(2\pi ft)dt$$

We did this integral in the last problem set – it's just $T/2$, or $1/2f$, where T is the period, since the average value of $\sin^2$ is ½ – so:

$$\Delta E = \frac{2}{3}\frac{1}{4\pi\epsilon_0}\frac{q^2}{c^3}(2\pi f)^4 A^2 \frac{1}{2f} = \frac{2}{3}\frac{1}{4\pi\epsilon_0}\frac{8\pi^4 f^3 A^2 q^2}{c^3}$$

The initial energy of the oscillator is simply $E_0 = \frac{1}{2}kA^2 = \frac{1}{2}m\omega_0^2 A^2$, using $\omega_0 = \sqrt{(k/m)}$, where $\omega_0 = 2\pi f$, given by the initial oscillation condition. Therefore:

$$\frac{\Delta E}{E_0} = \frac{2}{m4\pi^2 f^2 A^2}\frac{2}{3}\frac{1}{4\pi\epsilon_0}\frac{8\pi^4 f^3 A^2 q^2}{c^3} = \frac{2\pi f q^2}{3m\epsilon_0 c^3}.$$

It is instructive to plug in numbers – we'll return to this in a moment.

(b) We know that in general, the energy of a damped oscillator decays like

$E(t) = E_0 e^{-\gamma t} = E_0 e^{-\frac{\omega_0}{Q}t}$, using the definition of Q. Conceptually, we note the following: This E(t) expression can tell us how much energy is lost after one cycle. Our answer from (a) also tells us how much energy is lost after one cycle. **These two values must be the same!** Let's connect them. Assuming weak damping – we'll see in a moment if this is consistent – we can treat $\frac{\omega_0}{Q}t$ as small. The Taylor expansion of e^x is 1 + x + ..., so *E* after one period, i.e. $t = \frac{2\pi}{\omega_0}$ is $E(T) \approx E_0\left(1 - \frac{\omega_0}{Q}\frac{2\pi}{\omega_0}\right)$, so $\frac{\Delta E}{E_0} \approx \frac{2\pi}{Q}$. Combining this with our answer to part (a), we see that:

$$\frac{2\pi}{Q} = \frac{2\pi f q^2}{3m\epsilon_0 c^3}, \text{ and so } Q = \frac{3m\epsilon_0 c^3}{fq^2}$$

(c) Let us use $\lambda \approx 650 \times 10^{-9}$ m (hence the frequency $f = 4.61 \times 10^{14}$ Hz) and the charge and mass values given in the problem. (I forgot to write that $\epsilon_0 = 8.85 \times 10^{-12}$ m^{-3} kg^{-1} s^4 A^2, but this is easy to look up.)

Therefore $Q = \frac{3m\epsilon_0 c^3}{fq^2} = 5.5\times10^7$. **Q is very large, justifying our assumption of weak damping.**

(7) Experimenting with resonance. *Of course, answers will vary from person to person; take this as a guide. A good, brief response will include a description of the experimental system, a quantitative discussion of observations, and a comparison of observations with theoretical expectations.* – RP

I created a simple pendulum using keys hanging from a metal chain of length l = 49 cm. The keys are much heavier than the chain, approximating an "ideal" simple pendulum. I drove the pendulum by moving my hand, holding the chain, back and forth horizontally with varying frequencies. Using a clock, I measured the period of the driving force by measuring the time required for ten cycles of my hand's motion and dividing by ten. *Experimental setup – 3 pts.*

At low driving frequencies (e.g. a period of 2.0 s) the keys moved with a small amplitude, similar to that of my moving hand, and moved in phase with my hand. At high driving frequencies (e.g. a period of 0.5 s) the keys again moves with a small amplitude, but out of phase with my hand – i.e. moving "left" when my hand was moving "right." The maximal amplitude of the keys' oscillation occurred at a driving period of T_r = 1.3 seconds, or a period of $f = 1/T = 0.77$ Hz, which I will identify as resonance. At resonance, the amplitude of oscillation was so great that the chain would make angles of over 45° with the vertical. It was difficult to assess the phase of the response, but it seemed that my hand was moving most rapidly when the keys were at the end-points of their cycle, consistent with a 90 degree phase shift. *Observations – 3 pts.*

Given the length of the chain (stated above), I would expect a resonant frequency $f_{\text{expect}} = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{l}}$ = 0.71 Hz. This agrees well with the observed value of f = 0.77 Hz, given the precision of the experiment. *Comparison with theory / discussion – 3 pts.*