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University of Oregon; Fall 2007
Physics 351 – Vibrations and Waves

Problem Set 5: SOLUTIONS

(1) The bouncing man.

(a) We need to find ω_0, which we know from earlier work equals $\sqrt{(k/m)}$. What is the spring constant, k? Then the man is attached, the spring is extended by x = 20cm from its un-stretched equilibrium; at this point mg = kx, as we've seen before (problem set 2). Therefore $k = mg/x$ and $\omega_0 = \sqrt{k/m} = \sqrt{\frac{g}{x}}$.

Numerically, ω_0 = 7 radians/second.

(b) I drive the oscillator with a harmonic force F = $F_0 \cos(\omega t)$. What is F_0? I move my hand with an amplitude B, stretching and compressing the spring. The force conveyed by the spring is therefore kB, and so $F_0 = kB$. The *response* of the man is oscillation with an amplitude of A = 8 cm. We know that

$$A(\omega) = \frac{F_0}{k} \frac{\omega_0/\omega}{\sqrt{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}},$$

one of our many expressions for $A(\omega)$. At resonance ($\omega=\omega_0$), therefore, $A = \frac{F_0}{k} Q$. From above, $A = BQ$, and so Q=A/B= 8cm / 0.5cm = **Q=16**.

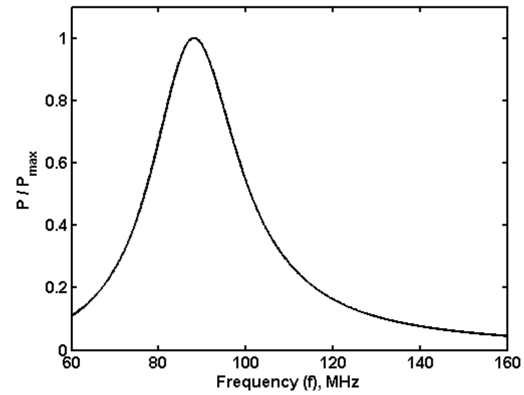
(c) The approximate $\bar{P}(\omega)$ relation $\bar{P}(\omega) = \frac{\gamma F_0^2}{2m} \frac{1}{\left[4(\omega_0 - \omega)^2 + \gamma^2\right]}$. We're concerned with $\omega=\alpha\omega_0$, where α=1.05. Rewriting, $\bar{P}(\omega) = \frac{\gamma F_0^2}{2m} \frac{1}{\omega_0^2\left[4(1-\alpha)^2 + \frac{1}{Q^2}\right]} = \frac{F_0^2}{2m\omega_0 Q} \frac{1}{\left[4(1-\alpha)^2 + \frac{1}{Q^2}\right]}$. Plugging in all the numbers, k=4.9 N/m, $F_0=k$B=0.0245 N, and $\bar{P}(\omega)$ **=0.002 Watts.** (You can check that the above expression is dimensionally correct.)

(2) Series RLC radio: As discussed in class and in the text, the maximum power absorption for a damped, driven oscillator occurs when the driving frequency (ω) equals the natural frequency (ω_0). (Keep in mind ω is the angular frequency, f is the "usual" frequency (cycles per unit time), and $\omega = 2\pi f$.) Also as discussed in class and in the text, $\omega_0 = \sqrt{1/(LC)}$ for a series RLC circuit.

(a) We want $\omega_0 = 2\pi$ 88.1 MHz, using C = 5 pF. Therefore we need an inductor $L = \left(C\omega_0^2\right)^{-1} = 6.5\times10^{-7} H$.

(b) The width (in frequency) of the resonance of a series RLC circuit, as discussed in class and in the text, is $\Delta f = (1/2\pi)\Delta\omega \approx R/L$, i.e. $(1/2\pi) \times 100\Omega\ /(0.65\times10^{-6})$H = 24 MHz. In other words, the circuit will pick up the desired 88.1 MHz signal, as well as every station near it in a 24 MHz window. As you might realize

looking at an FM dial, there are lots of stations in this range!
A plot of P(f), the power absorption at various frequencies:



(3) Overdamped oscillator motion. We know that the general solution to the motion of an overdamped oscillator is $x(t) = A_1 e^{-\left(\frac{\gamma}{2}+\beta\right)t} + A_2 e^{-\left(\frac{\gamma}{2}-\beta\right)t}$. We are told that *x*(t=0) =0 and *v*(t=0) = v_0. Therefore $x(t=0) = A_1 + A_2 = 0$, so $A_1 = -A_2$.

We could also note that the derivative $\dot{x}(t) = -A_1\left(\frac{\gamma}{2}+\beta\right)e^{-\left(\frac{\gamma}{2}+\beta\right)t} - A_2\left(\frac{\gamma}{2}-\beta\right)e^{-\left(\frac{\gamma}{2}-\beta\right)t}$,

so $\dot{x}(t=0) = -A_1\left(\frac{\gamma}{2}+\beta\right) - A_2\left(\frac{\gamma}{2}-\beta\right) = A_2\left(\frac{\gamma}{2}+\beta\right) - A_2\left(\frac{\gamma}{2}-\beta\right) = 2A_2\beta = v_0$, and thereby solve for A_2, but we'll see that this isn't even necessary for this problem.

Our particular solution $x(t) = A_1\left[e^{-\left(\frac{\gamma}{2}+\beta\right)t} - e^{-\left(\frac{\gamma}{2}-\beta\right)t}\right] = A_1 e^{-\frac{\gamma}{2}t}\left[e^{-\beta t} - e^{+\beta t}\right]$. Clearly, this is zero at t=0, as constructed. Is *x*(t)=0 anywhere else? The prefactor $e^{-\frac{\gamma}{2}t}$ is of course always positive (except at t $\rightarrow\infty$), so for *x*(t) to be zero, we need $e^{-\beta t} - e^{+\beta t} = 0$, i.e. $e^{-\beta t} = e^{+\beta t}$.
Taking logarithms, $-\beta t = \beta t$, which is satisfied only at t=0. So **no, *x*(t) is never zero except at t=0, therefore the door never hits Mr. K**.

4. Driven oscillator response.

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos\omega t.$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos\omega t.$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = F_0/m \cos\omega t.$$

then $$A(\omega) = \frac{F_0/m}{[(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2]^{1/2}}$$

(a) at $\omega = \omega_0$

$$A(\omega) = \frac{F_0/m}{\gamma^2\omega^2}$$

$$\therefore \frac{dA(\omega)}{d\omega} = -2\frac{F_0/m}{\gamma^2\omega^3} < 0$$

$\therefore$ at $\omega = \omega_0$, $A(\omega)$ is decreasing function.

(c) $$\tan\delta(\omega) = \frac{\gamma\omega}{\omega_0^2-\omega^2} = \frac{\omega\omega_0}{Q(\omega_0^2-\omega^2)}$$

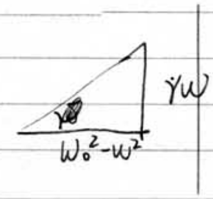
$$\therefore \frac{d\tan\delta(\omega)}{d\omega} = \frac{d\tan\delta}{d\delta}\frac{d\delta}{d\omega}$$

$$= \frac{1}{\cos^2\delta}\frac{d\delta}{d\omega} = \frac{d}{d\omega}\left(\frac{\omega\omega_0}{Q(\omega_0^2-\omega^2)}\right)$$

$$= \frac{\omega_0 Q(\omega_0^2-\omega^2) - \omega\omega_0 Q(-2\omega)}{Q^2(\omega_0^2-\omega^2)^2}$$

$$= \frac{\omega_0(\omega_0^2+\omega^2)}{Q(\omega_0^2-\omega^2)^2}.$$

$$\therefore \frac{d\delta}{d\omega} = \frac{\omega_0(\omega_0^2+\omega^2)}{Q(\omega_0^2-\omega^2)^2}\cos^2\delta$$



$$\cos\delta = \frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} \qquad \therefore \cos^2\delta = \frac{(\omega_0^2-\omega^2)^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

$$\therefore \frac{d\delta}{d\omega} = \frac{\omega_0(\omega_0^2+\omega^2)}{Q\cancel{(\omega_0^2-\omega^2)^2}} \cdot \frac{\cancel{(\omega_0^2-\omega^2)^2}}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} = \frac{\omega_0(\omega_0^2+\omega^2)}{Q[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}$$

at $\omega=\omega_0$

$$\frac{d\delta}{d\omega} = \frac{\omega_0 \cdot 2\omega_0^2}{Q\ \gamma^2\omega_0^2} = \frac{2\omega_0}{Q\cdot\frac{\omega_0^2}{Q^2}} = \frac{2Q}{\omega_0}$$

$\therefore$ at $\omega=\omega_0$, $\frac{d\delta}{d\omega}$ is an increasing function of Q

(5)

French 4-14.

(a) $$\overline{P}(\omega) = \frac{F_0^2\omega_0}{2kQ}\ \frac{1}{\left(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)^2+\frac{1}{Q^2}}$$

when $\omega = 0.98\,\omega_0$

$$\overline{P}(0.98\omega_0) = \frac{F_0^2\omega_0}{2kQ}\ \frac{1}{\left(\frac{1}{0.98}-0.98\right)^2+\frac{1}{Q^2}} = \frac{1}{2}\ \frac{F_0^2\omega_0}{2kQ}\cdot\frac{1}{\frac{1}{Q^2}}$$

that is $\left(\frac{1}{0.98}-0.98\right)^2+\frac{1}{Q^2} = \frac{2}{Q^2}$.

$$\therefore Q^2 = \frac{1}{\left(\frac{1}{0.98}-0.98\right)^2}$$

$$Q = \frac{1}{\frac{1}{0.98}-0.98} \doteq 24.75 \doteq 25$$

(b) $Q = \frac{\omega_0}{\gamma}$

$$\therefore \gamma = \frac{\omega_0}{Q} = \frac{\omega_0}{25} = 0.04\,\omega_0$$

(c) $$\frac{\Delta E}{E} = 1-e^{-\gamma T} \doteq 1-(1-\gamma T) = \gamma T = 0.04\omega_0\cdot\frac{2\pi}{\omega_0} = 0.08\pi \quad (Q\gg1)$$