Problem Set 5: SOLUTIONS

(1) The bouncing man.
(a) We need to find \( \omega_0 \), which we know from earlier work equals \( \sqrt{k/m} \). What is the spring constant, \( k \)? Then the man is attached, the spring is extended by \( x = 20 \text{cm} \) from its un-stretched equilibrium; at this point \( mg = kx \), as we’ve seen before (problem set 2). Therefore \( k = \frac{mg}{x} \) and \( \omega_0 = \sqrt{k/m} = \sqrt{\frac{g}{x}} \).

Numerically, \( \omega_0 = 7 \text{ radians/second} \).

(b) I drive the oscillator with a harmonic force \( F = F_0 \cos(\omega t) \). What is \( F_0 \)? I move my hand with an amplitude \( B \), stretching and compressing the spring. The force conveyed by the spring is therefore \( kB \), and so \( F_0 = kB \). The response of the man is oscillation with an amplitude of \( A = 8 \text{ cm} \). We know that

\[
A(\omega) = \frac{F_0}{k} \left( \frac{\omega_0}{\omega} \right)^{\frac{1}{2}} \left( 1 + \frac{1}{Q^2} \right),
\]

one of our many expressions for \( A(\omega) \). At resonance (\( \omega = \omega_0 \)), therefore, \( A = \frac{F_0}{k} Q \). From above, \( A = BQ \), and so \( Q = A/B = 8 \text{ cm} / 0.5 \text{ cm} = Q = 16 \).

(c) The approximate \( P(\omega) \) relation \( P(\omega) = \frac{\gamma F_0^2}{2m} \left( \frac{1}{(\omega_0 - \omega)^2 + \gamma^2} \right) \). We’re concerned with \( \omega = \alpha \omega_0 \), where \( \alpha = 1.05 \). Rewriting, \( P(\omega) = \frac{\gamma F_0^2}{2m} \left( \frac{1}{\omega_0^2 \left( 4 \left( 1 - \alpha \right)^2 + \frac{1}{Q^2} \right)} \right) = \frac{F_0^2}{2m \omega_0 Q} \left( \frac{1}{4 \left( 1 - \alpha \right)^2 + \frac{1}{Q^2}} \right) \). Plugging in all the numbers, \( k = 4.9 \text{ N/m} \), \( F_0 = kB = 0.0245 \text{ N} \), and \( P(\omega) = 0.002 \text{ Watts} \). (You can check that the above expression is dimensionally correct.)

(2) Series RLC radio: As discussed in class and in the text, the maximum power absorption for a damped, driven oscillator occurs when the driving frequency (\( \omega \)) equals the natural frequency (\( \omega_0 \)). (Keep in mind \( \omega \) is the angular frequency, \( f \) is the “usual” frequency (cycles per unit time), and \( \omega = 2\pi f \)). Also as discussed in class and in the text, \( \omega_0 = \sqrt{1/(LC)} \) for a series RLC circuit.

(a) We want \( \omega_0 = 2\pi 88.1 \text{ MHz} \), using \( C = 5 \text{ pF} \). Therefore we need an inductor \( L = \left( C \omega_0^2 \right)^{-1} = 6.5 \times 10^{-7} \text{ H} \).

(b) The width (in frequency) of the resonance of a series RLC circuit, as discussed in class and in the text, is \( \Delta f = (1/2\pi)\Delta \omega \approx R/L \), i.e. \( (1/2\pi) \times 100\Omega / (0.65 \times 10^{-6}) \Omega = 24 \text{ MHz} \). In other words, the circuit will pick up the desired 88.1 MHz signal, as well as every station near it in a 24 MHz window. As you might realize.
looking at an FM dial, there are lots of stations in this range!
A plot of $P(f)$, the power absorption at various frequencies:

![Plot of P(f)](image)

### (3) Overdamped oscillator motion.

We know that the general solution to the motion of an overdamped oscillator is $x(t) = A_1 e^{-\left(\frac{\gamma}{2}+\beta\right)t} + A_2 e^{-\left(\frac{\gamma}{2}-\beta\right)t}$. We are told that $x(t=0) = 0$ and $v(t=0) = v_0$. Therefore $x(t=0) = A_1 + A_2 = 0$, so $A_1 = -A_2$.

We could also note that the derivative $x'(t) = -A_1 \left(\frac{\gamma}{2} + \beta\right) e^{-\left(\frac{\gamma}{2}+\beta\right)t} - A_2 \left(\frac{\gamma}{2} - \beta\right) e^{-\left(\frac{\gamma}{2}-\beta\right)t}$, so $x'(t=0) = -A_1 \left(\frac{\gamma}{2} + \beta\right) - A_2 \left(\frac{\gamma}{2} - \beta\right) = 2A_2 \beta = v_0$, and thereby solve for $A_2$, but we’ll see that this isn’t even necessary for this problem.

Our particular solution $x(t) = A_1 \left[e^{-\left(\frac{\gamma}{2}+\beta\right)t} - e^{-\left(\frac{\gamma}{2}-\beta\right)t}\right] = A_1 e^{-\frac{\gamma}{2}t} \left[e^{-\beta t} - e^{\beta t}\right]$. Clearly, this is zero at $t=0$, as constructed. Is $x(t)=0$ anywhere else? The prefactor $e^{-\frac{\gamma}{2}t}$ is of course always positive (except at $t \to \infty$), so for $x(t)$ to be zero, we need $e^{-\beta t} - e^{\beta t} = 0$, i.e. $e^{-\beta t} = e^{\beta t}$.

Taking logarithms, $-\beta t = \beta t$, which is satisfied only at $t=0$. So no, $x(t)$ is never zero except at $t=0$, therefore the door never hits Mr. K.
4. Driven oscillator response.

\[m \ddot{x} + b \dot{x} + k x = F_0 \cos \omega t.\]
\[\ddot{x} + \frac{b}{m} \dot{x} + \frac{k}{m} x = \frac{F_0}{m} \cos \omega t.\]
\[\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t.\]

then \( A(\omega) = \frac{F_0/m}{\left(\omega_0^2 - \omega^2 \right)^2 + (\omega \omega_0)^2} \).

(a) at \( \omega = \omega_0 \)
\[A(\omega) = \frac{F_0/m}{\omega^2 \omega_0^2}\]
\[\frac{dA(\omega)}{d\omega} = -2 \frac{F_0/m}{\omega^2 \omega_0^2} < 0\]

\[\therefore \text{at } \omega = \omega_0, A(\omega) \text{ is decreasing function.}\]

(c) \[\tan \delta(\omega) = \frac{\omega \omega_0}{\omega_0^2 - \omega^2} = \frac{\omega \omega_0}{\omega_0^2 - \omega^2}\]

\[\frac{d\tan \delta(\omega)}{d\omega} = \frac{d\tan \delta}{d\omega} \frac{d\omega}{d\omega}\]
\[= \frac{1}{\cos^2 \delta} \frac{d\omega}{d\omega} = \frac{d}{d\omega} \left( \frac{\omega \omega_0}{\omega_0^2 - \omega^2} \right)\]
\[\delta = \frac{\omega_0 Q (\omega_0^2 - \omega^2) - \omega \omega_0 Q (-\omega)}{\lambda^2 (\omega_0^2 - \omega^2)^2}\]
\[= \frac{\omega_0 (\omega_0^2 + \omega^2)}{\lambda (\omega_0^2 - \omega^2)^2}\]

\[\therefore \frac{d\delta}{d\omega} = \frac{\omega_0}{\lambda (\omega_0^2 - \omega^2)^2} \cos^2 \delta\]
\[\cos \delta = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2}}\]
\[\therefore \cos^2 \delta = \frac{(\omega_0^2 - \omega^2)^2}{(\omega_0^2 - \omega^2)^2 + \omega^2} \omega^2\]
\[
\frac{d\delta}{dw} = \frac{u_0^2(w_0^2 + w^2)}{Q(w_0^2 - w^2)^2} \cdot \frac{(w_0^2 - w^2)^2}{(w_0^2 - w^2)^3 + 12w^2} = \frac{u_0^2(w_0^2 + w^2)}{Q[(w_0^2 - w^2)^3 + 12w^2]}
\]

At \( w = w_0 \),

\[
\frac{d\delta}{dw} = \frac{w_0 \cdot 2w^2}{Q \cdot y^2 w_0^2} = \frac{2w_0}{Q \cdot \frac{w_0^2}{Q^2}} = \frac{2Q}{w_0}
\]

\( w = w_0, \frac{d\delta}{dw} \) is an increasing function of \( Q \).

(5)

\[
\text{French 4.14.}
\]

(a) \( \bar{P}(w) = \frac{F_0^2 w_0}{2 \kappa Q} \cdot \frac{1}{(w_0^2 - w^2)^2 + \frac{1}{Q^2}} \)

When \( w = 0.98 w_0 \),

\[
\bar{P}(0.98 w_0) = \frac{F_0^2 w_0}{2 \kappa Q} \cdot \frac{1}{(0.98 - 0.98)^2 + \frac{1}{Q^2}} = \frac{1}{2} \cdot \frac{F_0^2 w_0}{2 \kappa Q} \cdot \frac{1}{\frac{1}{Q^2}}
\]

That is \( (\frac{1}{(0.98 - 0.98)^2} + \frac{1}{Q^2}) = 2 \cdot \frac{1}{Q^2} \).

\( \therefore \) \( Q^2 = \frac{1}{\frac{1}{Q^2} - 0.98} \)

\( Q = \frac{1}{\frac{1}{Q^2} - 0.98} \approx 24.75 \approx 25 \)

(b) \( \kappa = \frac{u_0^2}{y} \)

\( \therefore \) \( y = \frac{w_0}{\kappa} = \frac{w_0}{25} = 0.04 \; w_0 \)

(c) \( \frac{dE}{E} = e^{-\gamma T} = 1/(\gamma T) = \gamma T = 0.04 w_0 \cdot \frac{25}{w_0} = 0.08 \pi \) \((\pi \approx 1)\)