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University of Oregon; Fall 2007

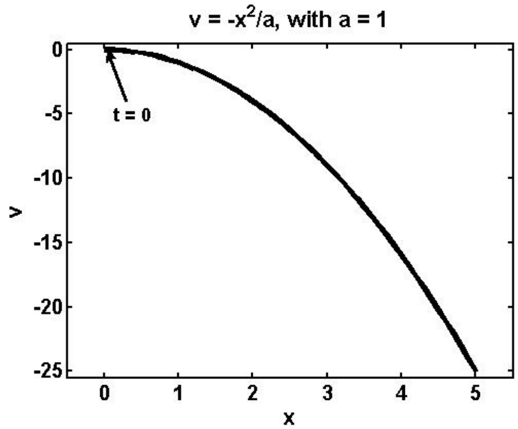
Physics 351 – Vibrations and Waves

Problem Set 6

Due date: Friday, Nov. 16, 4pm.
Reading: French Chapter 5.

A note: *This problem set is shorter than it seems. Problems 2-6 are very similar to one another, which should cement your understanding of coupled oscillators and should also be useful in analyzing the coupled oscillator systems that you build. Problem 1 is not related to coupled oscillators and could have been assigned weeks ago.*

(1, 6 pts.**) Phase space.** It's often informative to consider the trajectory of a system in "phase space," in which particular physical quantities form the coordinate axes. One common set of quantities is position and velocity. *For example*: let's imagine some system in which the position is given by $x(t) = a/t$, and the velocity is given by $v(t) = \dot{x} = -a/t^2$, where *t* is time and *a* is some constant. Therefore $v = -x^2/a$ – i.e. $v(x)$ is a quadratic function of *x*. At time *t*=0, both *v* and *x* are zero, so our parabola "starts" at the origin of our phase space plot. The plot looks like:

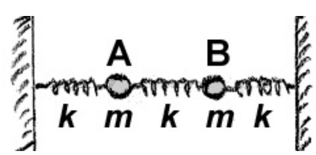


(a, 3 pts.**)** Consider an *undamped* simple harmonic oscillator. Prove that the phase space plot (with axes x and $\dot{x}$) is an ellipse. (Remind yourself of the equation characterizing an ellipse.)

(b, 2 pts.**)** Draw the phase space plot for an undamped oscillator with the initial conditions $x(t=0) = x_0$, $v(t=0) = 0$. Indicate the *t*=0 point on the plot.

(b, 1 pt.**)** What would the phase space plot of a damped oscillator look like? Draw it, qualitatively – you don't have to do any math. Indicate the "t=∞" point on the plot.

(2, 12 pts**) Two coupled masses.** Consider two objects (*A* and *B*) of equal mass *m* connected to each other and to rigid walls by identical springs of spring constant *k* (see Figure). Neglect damping, and assume all motion is in a horizontal plane, so gravity is irrelevant.



(a, 2 pts.**)** If one of the masses is clamped in place at its equilibrium position (i.e. *x* = 0 at all times), what is the angular frequency of oscillation of the other mass? (Call this ω_s). As usual, answering this involves writing the forces acting on the free mass, applying Newton's law to form a differential equation, and determining the oscillation frequency that satisfies the differential equation.

(**b**, 3 pts.**)** Now considering the unclamped system, defining $\omega_0^2 = k/m$, show that the system is described by two coupled differential equations:

$$(i)\quad \ddot{x}_A + \omega_s^2 x_A - \omega_o^2 x_B = 0$$

$$(ii)\quad \ddot{x}_B + \omega_s^2 x_B - \omega_o^2 x_A = 0$$

(**c**, 4 pts.**)** Find the normal mode frequencies. As we did in class for the coupled pendulums, write

$$\begin{aligned} x_A &= D_A \cos(\omega t - \phi) \\ x_B &= D_B \cos(\omega t - \phi) \end{aligned}, \quad \text{or} \quad \begin{aligned} x_A &= \mathrm{Re}\left[D_A e^{j(\omega t - \phi)}\right] \\ x_B &= \mathrm{Re}\left[D_B e^{j(\omega t - \phi)}\right] \end{aligned}$$

and substitute into (*i*) and (*ii*). You'll obtain two equations involving the frequencies and amplitudes. Write this pair of equations in matrix form, as

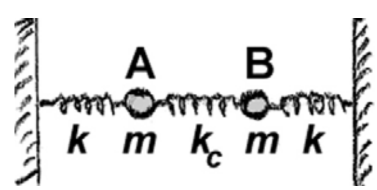
$$W \begin{bmatrix} D_A \\ D_B \end{bmatrix} = 0,$$

where W is a 2×2 matrix involving the frequencies. See the attached notes on matrix algebra if you are unfamiliar with matrix notation. A fundamental theorem of linear algebra states that this matrix equation has a non-trivial solution (i.e. a solution other than $D_A = D_B = 0$) if and only if the determinant of W is zero. See the attached notes on matrix algebra if you are unfamiliar with determinants. Find the values of ω for which det[W]=0. There will be two (ω_1 and ω_2). These are the normal mode frequencies. (Using the matrix determinant to determine ω is simpler than the approach taken in the text, as you may see from the subsequent problems.)

(**d,** 3 pts.**)** Find the normal mode amplitudes. For each normal mode frequency, substitute that ω in to one of Eqns. (*i*) or (*ii*) – it doesn't matter which, a consequence of the det[W]=0 condition – and determine the relation between D_A and D_B.

(**e,** 2 pts.**)** Write the general solution for $x_A(t)$ and $x_B(t)$. You may wish to call the amplitudes corresponding to the first mode "*C*" and the second mode "*D*,"as was done in the text for the two coupled pendulums (p. 125-126). Unlike the text's eq. 5-6, don't neglect the phase factors (i.e. ϕ, above).

(**3**, 8 pts**)** **Two coupled masses with non-identical springs.** Two identical masses (m), called A and B, are connected to rigid walls by springs of spring constant k, and to each other by a spring of spring constant k_c (see Figure).



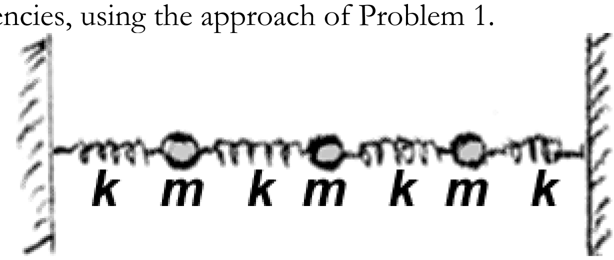
(**a**, 2 pts.**)** If one of the masses is clamped in place at its equilibrium position (i.e. $x = 0$ at all times), what is the angular frequency of oscillation of the other mass? (Call this ω_s).

(**b,** 4 pts.**)** Find the normal mode frequencies (for the unclamped system). Use the same approach as in Problem 2.

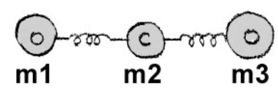
(**c,** 2 pts.**)** For small k_c (i.e. << k), show that the normal mode frequencies are given by the relation $\omega = \omega_s \pm \omega_0 \dfrac{k_c}{2k}$, to lowest order in $\dfrac{k_c}{k}$, where $\omega_0^2 = k/m$.

(4, 8 pts**) Two objects with an unknown coupling.** Consider two objects of mass *m* that, uncoupled, each exhibit oscillations with a period T_0. (*Note*: from this statement alone you should be able to write a differential equation of motion for each uncoupled oscillator, and state what the restoring "spring-like" force on each object is.) When coupled, object *A* feels an additional force equal to $-k_2 x_B$, where k_2 is some unknown coupling strength, and similarly, object *B* feels a force $-k_2 x_A$. The coupled system's two normal mode **periods** are measured to be T_1 (> T_0) and T_2 (< T_0). Determine the coupling strength k_2 in terms of the measured quantities T_0, T_1, and T_2.

(5, 8 pts**) Three coupled masses with identical springs.** Three identical masses (*m*) are connected to rigid walls and to each other by springs of spring constant *k* (see Figure). Determine the normal mode frequencies, using the approach of Problem 1.



(6, 10 pts.**) The CO_2 molecule (French 5-9)**. A carbon dioxide molecule can be modeled as two oxygen atoms (identical masses m_1 and m_3 in the figure) connected by identical springs of spring constant *k* to a carbon atom (of different mass, m_2). (The "springs," by the way, are related to the shape of the interaction potentials, as we know from our evaluation of the form of any potential energy function near equlibrium.)



(a, 8 pts.**)** Consider only motions in which the masses oscillate along the line joining their centers (i.e. "stretching" modes, rather than "bending" modes). Set up and solve the differential equations to determine the normal mode frequencies[1]. I recommend using the determinant method of Problem 1. *Hint*: The equation for m_3 is $m_3\ddot{x}_3 = -k(x_3 - x_2)$.

(b, 2 pt.**)** Noting that the mass of an oxygen atom is 16 a.m.u. and the mass of a carbon atom is 12 a.m.u., calculate the ratio of the normal mode frequencies. The stretching mode frequencies of CO_2 can easily be measured (e.g. by optical spectroscopy), and are found to be 40.0 and 70.5 THz (1 THz = 10^{12} Hz). Compare the ratio of these to the value you calculated. (Though molecules are governed by quantum, rather than classical, mechanics, classical calculations are often quite good!)

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[1] You'll find that there are only two normal mode frequencies, thought there are three masses. Why? Strictly speaking, the number of normal modes is not equal to *N*, the number of particles, but to the number of "degrees of freedom" of the system. In the other "three mass + springs" systems we considered, there are three degrees of freedom, the positions of each mass. Here, there is an additional constraint due to the absence of confining walls, not explicitly stated but nonetheless a consequence of Newton's laws: the center of mass can't move, since there is no external force acting on the system. The number of degrees of freedom for 1D motion is N minus the number of constraints, i.e. 2.