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Physics 351 – Vibrations and Waves

Problem Set 6: SOLUTIONS

(1) Phase Space.

Two methods:

(i) We know that for a simple harmonic oscillator, $x(t) = A\cos(\omega t - \phi)$ and $\dot{x} = -\omega A\sin(\omega t - \phi)$.

The equation for an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where *a* and *b* are the semimajor and semiminor axes.

(Recall that for a circle, $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$) where *a* is the radius of the circle.) Recall the simple trigonometric identity related to the unit circle, $\cos^2(y) + \sin^2(y) = 1$ for any *y*. We can write this trig identity in terms of our $x(t)$ and $\dot{x}(t)$:

$$\left(\frac{x(t)}{A}\right)^2 + \left(\frac{\dot{x}(t)}{-A\omega}\right)^2 = 1$$

which is an ellipse with $a=A$ and $b = A\omega$.

(ii) We know the equations that describe the position and velocity of an undamped harmonic oscillator as a function of time. We want to solve one of the equations for time and plug it into the other to get position and velocity in the same equation. Then we'll rearrange the equation to (hopefully) look like an ellipse. First we solve $x(t)$ (see above) for a harmonic oscillator for time.

$$t(x) = \frac{1}{\omega}\left[\arccos\left(\frac{x}{A}\right) + \phi\right]$$

Now we plug this result into the velocity equation.

$$\dot{x} = -\omega A\sin(\omega t - \phi) = -\omega A\sin(\omega\frac{1}{\omega}\left[\arccos\left(\frac{x}{A}\right) + \phi\right] - \phi)$$

$$= -\omega A\sin(\arccos(\frac{x}{A}) - \phi + \phi) = -\omega A\sin(\arccos\left(\frac{x}{A}\right));$$

Use trig identity (or draw triangles), $\sin(\arccos(x)) = \sqrt{1 - x^2}$

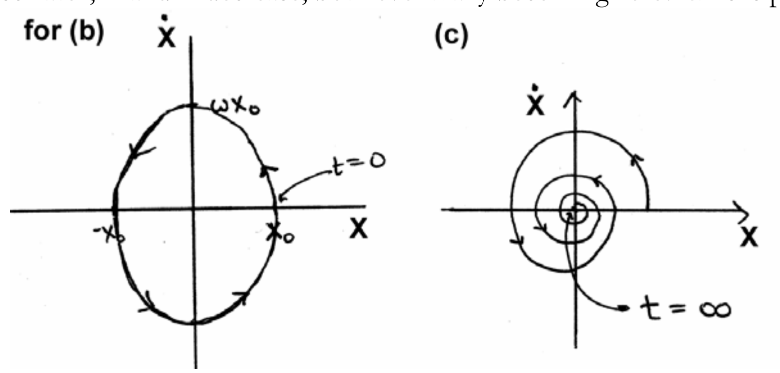
$$\dot{x} = -\omega A\sqrt{1 - \frac{x^2}{A^2}}$$

Therefore $\frac{\dot{x}^2}{\omega^2 A^2} = 1 - \frac{x^2}{A^2}$, or $1 = \frac{\dot{x}^2}{\omega^2 A^2} + \frac{x^2}{A^2}$

which results in an ellipse.

(b) For initial conditions $x(t=0)=x_0$, $v(t=0)=0$, $x(t)=x_0\cos(\omega t)$ and $\dot{x}=-\omega x_0\sin(\omega t)$, so the particular ellipse: $1=\dfrac{\dot{x}^2}{\omega^2 x_0^2}+\dfrac{x^2}{x_0^2}$. This is plotted below:

(c) For a damped oscillator, x and $\dot{x}$ decrease, both eventually becoming zero. This is plotted below:



Problem 2: Two coupled masses

a) If $x_B$is clamped then we have the following equation of motion:

$F = m\ddot{x}_A = -2k\,x_A$ or $m\ddot{x}_A + \omega_s^2 x_A = 0$ where $\omega_s^2 = \sqrt{\frac{2k}{m}}$.

b) The two differential equations describing this system are:

$m\ddot{x}_A = -k\,x_A + k(x_B - x_A)$ or $\ddot{x}_A + \frac{2k}{m}x_A - \frac{k}{m}x_B = 0$

$m\ddot{x}_B = -k\,x_B - k(x_B - x_A)$ or $\ddot{x}_B + \frac{2k}{m}x_B - \frac{k}{m}x_A = 0.$

Defining $\omega_0^2 = \frac{k}{m}$ and taking ω_s^2 from part a) we have:

$\ddot{x}_A + \omega_s^2 x_A - \omega_0^2 x_B = 0$

$\ddot{x}_B + \omega_s^2 x_B - \omega_0^2 x_A = 0.$

(c) Assume normal mode solutions: $\begin{array}{l} x_A = D_A\cos(\omega t-\phi) \\ x_B = D_B\cos(\omega t-\phi) \end{array}$. Differentiating and "plugging in" to our differential equations gives

$$-\omega^2 D_A + \omega_s^2 D_A - \omega_0^2 D_B = 0$$
$$-\omega^2 D_B + \omega_s^2 D_B - \omega_0^2 D_A = 0$$

which leads to the matrix equation

$$\begin{pmatrix} -\omega^2+\omega_s^2 & -\omega_0^2 \\ -\omega_0^2 & -\omega^2+\omega_s^2 \end{pmatrix}\begin{pmatrix} D_A \\ D_B \end{pmatrix} = 0$$

The non-trivial solution comes from setting the determinant of the 2x2 matrix equal to zero:

$$\begin{vmatrix} -\omega^2+\omega_s^2 & -\omega_0^2 \\ -\omega_0^2 & -\omega^2+\omega_s^2 \end{vmatrix} = \left(-\omega^2+\omega_s^2\right)^2 - \left(-\omega_0^2\right)^2 = 0$$

$$\left(-\omega^2+\omega_s^2\right) = \pm\omega_0^2$$

$$\omega^2 = \omega_s^2 \pm \omega_0^2$$

So $\boxed{\omega = \frac{3k}{m} \text{ or } \frac{k}{m}}$

(d) Taking $\omega = \sqrt{\frac{k}{m}} = \omega_0$ and plugging it into the first row of our matrix equation gives

$$-\omega_0^2 D_A + \omega_s^2 D_A - {\omega_0}^2 D_B = 0$$

$$\left(-\omega_0^2+\omega_s^2\right) D_A - {\omega_0}^2 D_B = 0$$

$$-\omega_0^2 D_A - {\omega_0}^2 D_B = 0 \quad \text{using the } \omega \text{ expressions}$$

$$D_A = D_B$$

Next taking $\omega = \sqrt{\frac{3k}{m}}$ and doing similar algebra:

$$-\frac{k}{m} D_A - \frac{k}{m} D_B = 0$$

$$D_A = -D_B$$

(e) We have two normal mode frequencies that yield the following general solution:

We have two normal mode frequencies that yield to following general solution:

$$x_A(t) = C\cos\left(\sqrt{\frac{k}{m}}\ t+\alpha\right) + D\cos\left(\sqrt{\frac{3k}{m}}\ t+\beta\right)$$

$$x_B(t) = C\cos\left(\sqrt{\frac{k}{m}}\ t+\alpha\right) - D\cos\left(\sqrt{\frac{3k}{m}}\ t+\beta\right).$$

(3) Two coupled masses with non-identical springs.

a) If B is clamped in place then we have an equation of motion similar to part a) of problem **2**

$F = m\ddot{x}_A = -k\,x_A - k_c\,x_A$ or $\ddot{x}_A + \frac{(k+k_c)}{m}\,x_A = \ddot{x}_A + \omega_s^2\,x_A = 0$ where $\omega_s^2 = \frac{(k+k_c)}{m}$.

b) We have two equations of motion one for each mass:

$m\ddot{x}_A = -k\,x_A + k_c(x_B - x_A)$ or $\ddot{x}_A + \omega_s^2\,x_A - \omega_c^2\,x_B = 0$ where $\omega_c^2 = \frac{k_c}{m}$.

$m\ddot{x}_B = -k\,x_B - k_c(x_B - x_A)$ or $\ddot{x}_B + \omega_s^2\,x_B - \omega_c^2\,x_A = 0$.

Let's assume the solutions:

$x_A = C_A\cos(\omega t + \alpha)$

$x_B = C_B\cos(\omega t + \alpha)$.

Plugging these in gives us:

$-\omega^2 C_A + \omega_s^2 C_A - \omega_c^2 C_B = 0$

$-\omega^2 C_B + \omega_s^2 C_B - \omega_c^2 C_A = 0$.

And the matrix equation:

$$\begin{pmatrix} -\omega^2 + \omega_s^2 & -\omega_c^2 \\ -\omega_c^2 & -\omega^2 + \omega_s^2 \end{pmatrix} \begin{pmatrix} C_A \\ C_B \end{pmatrix} = 0.$$

Once again we take the determinant of the 2x2 matrix and set it equal to zero to find the normal mode frequencies ω.

$$\begin{vmatrix} -\omega^2 + \omega_s^2 & -\omega_c^2 \\ -\omega_c^2 & -\omega^2 + \omega_s^2 \end{vmatrix} = (-\omega^2 + \omega_s^2)^2 - \omega_c^4 = 0.$$

Solving this for ω gives us:

$\omega^2 = \omega_s^2 \pm \omega_c^2 = \frac{k+2k_c}{m}$ or $\frac{k}{m}$.

c) As the hint suggests let's begin by factoring out an ω_s:

$\omega = \omega_s\sqrt{1 \pm \left(\frac{\omega_c}{\omega_s}\right)^2} = \omega_s\sqrt{1 \pm \left(\frac{k_c}{k+k_c}\right)} = \omega_s\sqrt{\frac{k+k_c \pm k_c}{k+k_c}} = \omega_s\sqrt{\frac{1+r\pm r}{1+r}}$ where $r = \frac{k_c}{k} << 1$. Now we need to Taylor expand the square root term to extract the behavior of the complicated term for lowest order in r. Evaluating the plus sign first and keeping the lowest order r terms we have:

$\sqrt{\frac{1+r+r}{1+r}} \approx 1 + \frac{r}{2} + \cdots$

Similarly for the minus sign we have:

$\sqrt{\frac{1+r-r}{1+r}} \approx 1 - \frac{r}{2} + \cdots$

Putting these two results together we have:

$\omega = \omega_s\sqrt{\frac{1+r\pm r}{1+r}} \approx \omega_s\left(1 \pm \frac{r}{2}\right) = \omega_s\left(1 \pm \frac{k_c}{2k}\right)$.

(You could also Taylor expand the "original" $\omega = \omega_s \sqrt{\frac{1+r \pm r}{1+r}}$ expression and keep the lowest-order term in r – you'd get the same answer.)

(4) Two objects with an unknown coupling.

We start this problem by recognizing that the uncoupled equations of motion are:

$$m\,\ddot{x}_A = -k\,x_A \text{ or } \ddot{x}_A + \omega_0^2\,x_A = 0$$

$$m\,\ddot{x}_B = -k\,x_B \text{ or } \ddot{x}_B + \omega_0^2\,x_B = 0 \text{ where } \omega_0 = \frac{2\pi}{T_0} \text{ giving us } k = m\,\omega_0^2 = m\left(\frac{2\pi}{T_0}\right)^2.$$

We are told how the coupling changes each equation of motion (adding a $-k_2\,x_i$ force term). Taking into account the coupling we now have:

$$\ddot{x}_A + \omega_0^2\,x_A - \omega_2^2\,x_B = 0$$

$$\ddot{x}_B + \omega_0^2\,x_B - \omega_2^2\,x_A = 0 \text{ where } \omega_2^2 = \frac{k_2}{m}.$$

Now we proceed as we did in the previous two problems and assume the following solutions:

$$x_A = C_A\cos(\omega\,t + \alpha)$$

$$x_B = C_B\cos(\omega\,t + \alpha).$$

Substituting these solutions into our differential equations gives us:

$$-\omega^2\,C_A + \omega_0^2\,C_A - \omega_2^2\,C_B = 0$$

$$-\omega^2\,C_B + \omega_0^2\,C_B - \omega_2^2\,C_A = 0.$$

Taking this to matrix form we have:

$$\begin{pmatrix} \omega_0^2 - \omega^2 & -\omega_2^2 \\ -\omega_2^2 & \omega_0^2 - \omega^2 \end{pmatrix} \begin{pmatrix} C_A \\ C_B \end{pmatrix} = 0.$$

Following our tried and true strategy we take the determinant of the 2x2 matrix and set it equal to zero to find the normal mode frequencies.

$$\begin{vmatrix} \omega_0^2 - \omega^2 & -\omega_2^2 \\ -\omega_2^2 & \omega_0^2 - \omega^2 \end{vmatrix} = (\omega_0^2 - \omega^2)^2 - \omega_2^2 = 0.$$

Solving for ω yields:

$$\omega^2 = \omega_0^2 \pm \omega_2^2 = \frac{k \pm k_2}{m}.$$

However, what we're after is ω_2 so we want:

$$\omega_2^2 = \omega^2 - \omega_0^2 \text{ or } \omega_0^2 - \omega^2.$$

Now recall the period of oscillations for the coupled system are $T_1 > T_0$ and $T_2 < T_0$. So for T_1 we have $\omega_0 > \omega$ which allows us to solve for ω_2:

$$\omega_2^2 = \omega_0^2 - \omega^2 = \left(\frac{2\pi}{T_0}\right)^2 - \left(\frac{2\pi}{T_1}\right)^2 \text{ and } k_2 = m\,\omega_2^2 = m\left(\frac{2\pi}{T_0}\right)^2 - \left(\frac{2\pi}{T_1}\right)^2.$$

Similarly, for $T_2 < T_0$ we have $\omega > \omega_0$ giving us:

$$\omega_2^2 = \omega^2 - \omega_0^2 = \left(\frac{2\pi}{T_2}\right)^2 - \left(\frac{2\pi}{T_0}\right)^2 \text{ and } k_2 = m\,\omega_2^2 = m\left(\frac{2\pi}{T_2}\right)^2 - \left(\frac{2\pi}{T_0}\right)^2.$$

(5) Three coupled masses with identical springs.

Following the lead of problem 6.2 we write down the coupled equations of motion for this system:

$$\begin{aligned} 0 &= \ddot{x}_A + 2\omega_0^2 x_A - \omega_0^2 x_B && (1) \\ 0 &= \ddot{x}_B + 2\omega_0^2 x_B - \omega_0^2 x_A - \omega_0^2 x_C && (2) \\ 0 &= \ddot{x}_C + 2\omega_0^2 x_C - \omega_0^2 x_B, && (3) \end{aligned}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$. We again assume solutions of the form $x_i(t) = C_i \cos(\omega t + \alpha)$ and plug them into the above equations:

$$\begin{aligned} 0 &= -\omega^2 C_A + 2\omega_0^2 C_A - \omega_0^2 C_B && (4) \\ 0 &= -\omega^2 C_B + 2\omega_0^2 C_B - \omega_0^2 C_A - \omega_0^2 C_C && (5) \\ 0 &= -\omega^2 C_C + 2\omega_0^2 C_C - \omega_0^2 C_B. && (6) \end{aligned}$$

Putting this into matrix form we have:

$$\begin{pmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{pmatrix} \begin{pmatrix} C_A \\ C_B \\ C_C \end{pmatrix} = 0$$

Now we proceed as before and recognize that solving the determinant of the above matrix set equal to zero for ω will give us the normal mode frequencies. This gives us

$$(2\omega_0^2 - \omega^2)\left[(2\omega_0^2 - \omega^2)^2 - \omega_0^4\right] - (-\omega_0^2)\left[-\omega_0^2(2\omega_0^2 - \omega^2)\right] = 0$$

One could multiply out all the terms and simplify, getting:

$$-\omega^6 + 6\omega^4\omega_0^2 - 10\omega^2\omega_0^2 + 4\omega_0^6 = 0.$$

But it is easier to first look at our expression, and see what we can factor out. Notice that there's a $(2\omega_0^2 - \omega^2)$ in common to both terms. Therefore

$$2\omega_0^2 - \omega^2 = 0,$$

or $\omega = \sqrt{2}\omega_0$, is one solution, i.e. one normal mode frequency. (The negative root is physically identical, and so irrelevant.) For $\omega \neq \sqrt{2}\omega_0$, we can divide through by $2\omega_0^2 - \omega^2$, getting:

$$(2\omega_0^2 - \omega^2)^2 - \omega_0^4 - \omega_0^4 = 0$$

for the other normal mode frequencies, which gives

$$2\omega_0^2 - \omega^2 = \pm\sqrt{2}\omega_0^2,$$

or $\omega^2 = (2 \pm \sqrt{2})\omega_0^2$.

The three normal mode frequencies, therefore, are

$$\omega = \sqrt{2}\omega_0, \omega_0\sqrt{2 - \sqrt{2}}, \omega_0\sqrt{2 + \sqrt{2}}$$