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University of Oregon; Fall 2007

Physics 351 – Vibrations and Waves

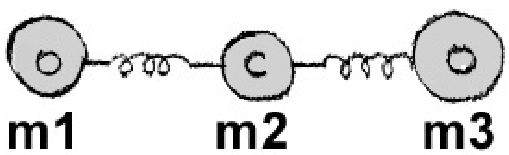
Problem Set 7

Due date: Wednesday, Nov. 21, 5pm. *No late homework will be accepted, due to the Thanksgiving break.*

Reading: French Chapters 5-6.

***Note*:** This problem set is shorter than it looks. Still, don't wait until the last minute!

(1, 9 pts.**) The CO_2 molecule (French 5-9)**. [*Note: this problem was originally problem 6 of PS6. If you turned it in last week, please note on this problem set that you did so, as a reminder to us.*] A carbon dioxide molecule can be modeled as two oxygen atoms (identical masses m_1 and m_3 in the figure) connected by identical springs of spring constant *k* to a carbon atom (of different mass, m_2). (The "springs," by the way, are determined by the form of the interaction potentials, as we know from earlier analyses.)



(a, 7 pts.**)** Consider only motions in which the masses oscillate along the line joining their centers (known as "stretching" modes, rather than "bending" modes). Set up and solve the differential equations to determine the normal mode frequencies[1]. I recommend using the determinant method of Problem 2 of Problem Set 6. *Hint*: The equation for m_3 is $m_3\ddot{x}_3 = -k(x_3 - x_2)$.

(b, 2 pt.**)** Noting that the mass of an oxygen atom is 16 a.m.u. and the mass of a carbon atom is 12 a.m.u., calculate the ratio of the normal mode frequencies. The stretching mode frequencies of CO_2 can easily be measured (e.g. by optical spectroscopy), and are found to be 40.0 and 70.5 THz (1 THz = 10^{12} Hz). Compare the ratio of these to the value you calculated. (Though molecules are governed by quantum, rather than classical, mechanics, classical calculations are often quite good!)

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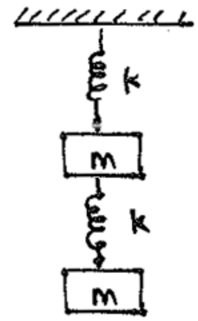
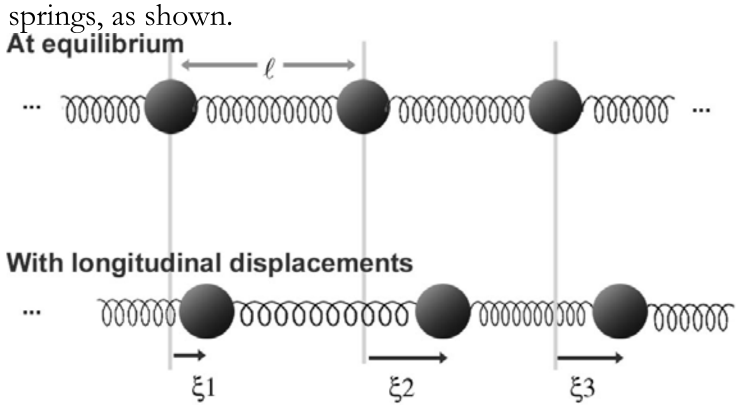
[1] You'll find that there are only two normal mode frequencies, thought there are three masses. Why? Strictly speaking, the number of normal modes is <u>not</u> equal to *N*, the number of particles, but to the number of "degrees of freedom" of the system. In the other "three mass + springs" systems we considered, there are three degrees of freedom, the positions of each mass. Here, there is an additional constraint due to the absence of confining walls, not explicitly stated but nonetheless a consequence of Newton's laws: the center of mass of the system can't move, since there is no external force acting on the system. The number of degrees of freedom for 1D motion is *N* minus the number of constraints, i.e. 2.

(2, 5 pts.) Hanging springs. Consider two masses (each of mass *m*) hanging from identical springs (of spring constant *k*) in a uniform gravitational field (of acceleration *g*) as shown. Determine the normal mode frequencies of this coupled oscillator system. (*Hint #1*: Convince yourself that gravity is "irrelevant" – it affects the equilibrium positions of the masses, from which you should define your coordinate system, but doesn't provide any position-dependent force. *Hint #2:* The product of the normal mode frequencies will turn out to be $\frac{k}{m}$.)



(3, 4 pts.**)** **Longitudinal oscillations of *N* spring-coupled masses.** Let's consider a system of N masses (each of mass m) connected by identical springs (k). See figure. At equilibrium, each mass is separated from its neighbor by distance ℓ, and the total length, $L = (N+1)\ell$, is kept constant (e.g. by pinning the masses at the ends to rigid walls). We'll consider longitudinal oscillations – i.e. motions along the axis of the springs, as shown.



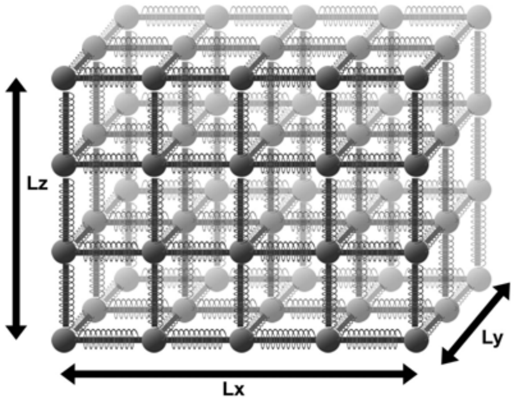
(a, 2 pts.**)** Consider mass #*p*. Write down the forces acting on mass *p* when it is displaced from equilibrium, and via Newton's laws derive the equation of motion:

$$\ddot{\xi}_p + 2\omega_0^2 \xi_p - \omega_0^2 \left(\xi_{p+1} + \xi_{p-1}\right), \text{ where } \omega_0^2 = k/m.$$

(b, 2 pts.**)** The above equation looks <u>exactly</u> like the equation we derived for the transverse displacements (y_p) of the loaded string, and so we already "know" the normal mode frequencies. For large N, and $n << N$, show that:

$$\omega_n \approx \frac{n\pi}{L}\sqrt{\frac{kl^2}{m}}.$$

(4, 11 pts.**)** **Crystal vibrations.** The oscillations described in Problem 3 are a good model for the vibrations of atoms in a crystal. How can we relate the normal mode frequencies to the "macroscopic" properties of a solid (i.e. things like density, which are easily measured, rather than ℓ or k, which are not)? Consider a cubic lattice of atoms (each of mass m) held together by springs (k) (see Figure, below). At equilibrium, the spacing between adjacent atoms is ℓ. The lattice has lengths L_x, L_y, and L_z in the x, y, and z directions.



(**a**, 1 pt.**)** Express the density of this solid in terms of m and ℓ. (Consider the solid to be very large – i.e. don't worry about the edges – so that the answer is very simple.)

(**b**, 2 pts.**)** Imagine applying a force F, in the $+x$ direction, to <u>each</u> of the springs <u>at the $+x$ face</u> of the lattice – i.e. stretching the lattice by pulling one face. Explain why the force felt by <u>any</u> mass in the solid, at equilibrium, is equal to F. *Hint*: Start by thinking about the mass immediately adjacent to the one at the end that's being pulled. What force, at equilibrium, does the end-mass exert on it?

(**c**, 1 pt.**)** Define ξ as the displacement of a single spring from equilibrium due to applied force F. Given the application of force described in (*b*), show that the fractional extension of the solid, $\Delta L/L$, equals ξ/ℓ.

(**d**, 2 pts.**)** Given the definition of the Young's modulus Y (see p. 46-47 of the text) as the ratio of the stress (force per unit area) to strain (fractional displacement), show that $Y = k/\ell$.

(**e,** 2 pts.) Show (using the result from Problem 3) that for large N and $n << N$, $\omega_n \approx \frac{n\pi}{L}\sqrt{\frac{Y}{\rho}}$.

This gives us the normal mode frequencies in terms of easily measured parameters.

(**f,** 2 pts.**)** Show that $\left(Y/\rho\right)^{1/2}$ has dimensions of speed, and that it has the same dimensions as $\left(\tau/\mu\right)^{1/2}$, the quantity that appeared in the expression we derived for the normal mode frequencies of a loaded string. (τ is the tension in the string, and μ the mass density.) By the way, this is the speed of sound in the solid.

(**g,** 1 pt.**)** What is the <u>period</u> of oscillation for the lowest-frequency normal mode of a $(1\ \text{cm})^3$ **quartz** crystal? The density of quartz is about 2.65 g/cm^3; its Young's Modulus is about 85×10^9 N/m^2.

(**h,** 1 pt.**)** What is the <u>period</u> of oscillation for the lowest-frequency normal mode of a $(1\ \text{cm})^3$ block of **Jello**[2]? The density of jello is about the same as water (since it's mostly water), and its Young's Modulus is about 100 N/m^2. Feel free to test this by poking some Jello.

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[2] Jello isn't a crystal, but that doesn't really matter. More significantly, Jello isn't a simple solid but a mysterious "complex fluid" that doesn't have a well-defined Young's modulus. We'll ignore this here.

(5, 4 pts.**) Banjo strings.** The fundamental frequencies f_1 (i.e. n=1 normal mode frequencies) of the strings of a five-string banjo, their lengths, and the tensions to which the strings are stretched are given in the table below. (Note that I've written the frequencies, f, not angular frequencies, $\omega=2\pi f$.)

string	**#1 (D)**	**#2 (B)**	**#3 (G)**	**#4 (D)**	**#5 (G)**
f_1 (Hz)	293	247	196	147	392
Tension (N)	48.64	42.49	45.56	60.61	48.69
Length (cm)	68.6	68.6	68.6	68.6	51.3

(a, 2 pts.**)** Which two of the strings have the same mass density?
(b, 2 pts.**)** Mr. K. decides to revolutionize the world of bluegrass music by using only the sort of string material identified in Part (a) for all five banjo strings. He'll leave the string lengths unchanged from their standard values. What tension values should he use for the five strings? (I don't recommend this approach to banjo design, by the way.)

(6, 2 pts.**) Plucking a string.** Plucking a string, like a guitar string, leads to vibrations at many normal modes, not just the "fundamental" (n=1) mode. Plucking a sting near its center, one finds that modes with odd n are excited much more than even n modes. Why? *Hint*: Sketch what the modes look like. You don't have to do any calculations.
By the way: you can hear the difference between plucking a guitar string at two different locations via an applet at: http://www.bsharp.org/physics/stuff/GuitarA.html

(6) Work on your **coupled oscillator project!**