Physics 351, Fall 2007, Prof. Parthasarathy
Problem Set 7 Solutions

1. CO_2:

   (a) This problem requires the same treatment as PS6 no. 2 but with a different set of differential equations. The forces acting on masses 1, 2, and 3 are:

$$\begin{aligned} F_1 &= -k(x_1 - x_2) && (1) \\ F_2 &= -k(x_2 - x_1) - k(x_2 - x_3) && (2) \\ F_3 &= -k(x_3 - x_2), && (3) \end{aligned}$$

and these equal $m\ddot{x}_{1,2,3}$ as usual. The coupled equations of motion are then:

$$\begin{aligned} 0 &= \ddot{x}_1 + \omega_1^2 x_1 - \omega_1^2 x_2 && (4) \\ 0 &= \ddot{x}_2 + 2\omega_2^2 x_2 - \omega_2^2 x_1 - \omega_2^2 x_3 && (5) \\ 0 &= \ddot{x}_3 + \omega_3^2 x_3 - \omega_3^2 x_2, && (6) \end{aligned}$$

where $\omega_i = \sqrt{\frac{k}{m_i}}$. Assuming solutions of the form $x_i(t) = C_i \cos(\omega t + \alpha)$ and plugging them into the above equations:

$$\begin{aligned} 0 &= -\omega^2 C_1 + \omega_1^2 C_1 - \omega_1^2 C_2 && (7) \\ 0 &= -\omega^2 C_2 + 2\omega_2^2 C_2 - \omega_2^2 C_1 - \omega_2^2 C_3 && (8) \\ 0 &= -\omega^2 C_3 + \omega_3^2 C_3 - \omega_3^2 C_2. && (9) \end{aligned}$$

Which then gives us the following matrix equation:

$$\begin{pmatrix} \omega_1^2 - \omega^2 & -\omega_1^2 & 0 \\ -\omega_2^2 & 2\omega_2^2 - \omega^2 & -\omega_2^2 \\ 0 & -\omega_3^2 & \omega_3^2 - \omega^2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = 0.$$

Taking the determinant of the matrix and setting equal to zero gives us:

$$(\omega_1^2 - \omega^2)\left[(2\omega_2^2 - \omega^2)(\omega_3^2 - \omega^2) - \omega_2^2\omega_3^2\right] + \omega_1^2\left[-\omega_2^2(\omega_3^2 - \omega^2)\right] = 0.$$

Recall that $m_1 = m_3$ so $\omega_1 = \omega_3$ which we'll call ω_{13}. Making this substitution allow us to factor out $(\omega_{13}^2 - \omega^2)$:

$$(\omega_{13}^2 - \omega^2)\left[-\omega^2\omega_{13}^2 - 2\omega^2\omega_2^2 + \omega^4\right] = 0.$$

There are two routes to a solution – either of these factors can be zero. The left is especially easy to examine. We see that $(\omega_{13}^2 - \omega^2) = 0$, i.e. $\omega = \omega_{13}$, is one normal mode solution; factoring this out we have

$$\begin{aligned} 0 &= \left[-\omega^2\omega_{13}^2 - 2\omega^2\omega_2^2 + \omega^4\right] && (10) \\ &= \omega^2\left[\omega^2 - (\omega_{13}^2 + 2\omega_2^2)\right] && (11) \end{aligned}$$

giving us $\omega = 0$ (an uninteresting, non-oscillatory, stationary solution) or $\omega^2 = (\omega_{13}^2 + 2\omega_2^2)$.

The two normal mode frequencies are therefore:

$$\omega = \omega_{13} = \sqrt{\frac{k}{m_1}} \text{ and } \sqrt{\omega_{13}^2 + 2\omega_2^2} = \sqrt{\frac{k}{m_1} + \frac{2k}{m_2}}$$

(b) The ratio of the frequencies is:

$$R = \sqrt{\frac{\frac{k}{m_1} + \frac{2k}{m_2}}{\frac{k}{m_1}}} = \sqrt{1 + 2\frac{m_1}{m_2}}.$$

Plugging in the given values of $m_1 = 16, m_2 = 12$ we have,

$$R = 1.91.$$

Observed values of oscillation frequencies give us:

$$R = \frac{70.5}{40.0} = 1.76.$$

Pretty close.

2. Hanging springs: Defining x_A and x_B as the displacements *from equilibrium* of each of the two masses, the forces acting on masses A and B are:

$$A: \quad -kx_A + k(x_B - x_A) \tag{12}$$
$$B: \quad -k(x_B - x_A) \tag{13}$$

As usual, it's convenient to derive this by considering each spring separately. (Note that this has the correct behavior for increasing x_A or x_B.) Therefore our differential equations of motion are, setting the forces equal to $m\ddot{x}$ and defining $\omega_0^2 = k/m$,

$$A: \quad \ddot{x_A} + 2\omega_0^2 x_A - \omega_0^2 x_B = 0 \tag{14}$$
$$B: \quad \ddot{x_B} + \omega_0^2 x_B - \omega_0^2 x_A = 0 \tag{15}$$

As usual (see PS6), we look for a normal mode solution $x_A = C_A \cos(\omega t)$, $x_B = C_B \cos(\omega t)$:

$$A: \quad (-\omega^2 + 2\omega_0^2)C_A - \omega_0^2 C_B = 0 \tag{16}$$
$$B: \quad (-\omega^2 + \omega_0^2)C_B - \omega_0^2 C_A = 0 \tag{17}$$

which in matrix form is

$$\begin{pmatrix} (-\omega^2 + 2\omega_0^2) & -\omega_0^2 \\ -\omega_0^2 & (-\omega^2 + \omega_0^2) \end{pmatrix} \begin{pmatrix} C_A \\ C_B \end{pmatrix} = 0.$$

Taking the determinant of the matrix and setting it equal to zero to find non-trivial solutions gives:

$$(-\omega^2 + 2\omega_0^2)(-\omega^2 + \omega_0^2)^{-}(\omega_0^2)^2 = 0 \tag{18}$$

$$(-\omega^2 + 2\omega_0^2) = \pm\omega_0^2 \tag{19}$$

$$\omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 = 0 \tag{20}$$

We solve for ω^2 using the quadratic formula, from which:

$$\omega^2 = \frac{3\omega_0^2 \pm \sqrt{9\omega_0^4 - 4\omega_0^4}}{2} \tag{21}$$

$$\omega^2 = \omega_0^2 \frac{1}{2}(3 \pm \sqrt{5}) \tag{22}$$

$$= \frac{k}{2m}(3 \pm \sqrt{5}) \tag{23}$$

These are the two normal mode frequencies. (You can verify that their product is k/m, as promised.)

3. N coupled masses:

   (a) Let's write down the sum of all forces on mass p.

   $$m\ddot{\xi}_p = -k(\xi_p - \xi_{p-1}) + k(\xi_{p+1} - \xi_p)$$

   which is just like the equation of motion for the middle mass of the previous problem. This should make sense since the two situations (regarding the middle mass) are identical. Now let $\omega_0^2 = \frac{k}{m}$. Then,

   $$\ddot{\xi}_p + 2\omega_0^2\xi_p - \omega_0^2(\xi_{p-1} + \xi_{p+1} = 0.$$

   (b) The normal mode frequencies that correspond to the above equation are:

   $$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right).$$

   Let's consider $n << N$. Then, $\frac{n\pi}{2(N+1)}$ is a small number and we can make a small angle approximation.

   $$\omega_n \approx 2\omega_0 \left(\frac{n\pi}{2(N+1)}\right).$$

   Recall that in the problem statement we made the assertion that $L = (N+1)\ell$ that relates the total length of the coupled mass chain to the length of one spring. Using this expression in the form $(N+1) = \frac{L}{\ell}$ and $\omega_0 = \sqrt{\frac{k}{m}}$ we have,

   $$\omega_n \approx \sqrt{\frac{k\ell^2}{m}}\left(\frac{n\pi}{L}\right).$$

4. Crystal vibrations:

(a) The definition of density is total mass divided by total volume. The total mass of our lattice is $M = m(N_x N_y N_z)$. The total volume of our lattice is,

$$V = L_x L_y L_z = \ell^3(N_x - 1)(N_y - 1)(N_z - 1).$$

But we're looking at large N_i so $N_i - 1 \approx N_i$. Putting this together we have,

$$\rho = \frac{M}{V} = \frac{m}{\ell^3}$$

(b) When we act on the end mass with a force F the mass will be displaced a distance ξ. The spring will then be compressed by the same amount ξ. The spring has constant k so the restoring force felt by the next mass in the chain is $k\xi$. But as we initially stated the original force F corresponds to a compression ξ for spring constant k. So the force felt by the next mass is indeed F. This force will result in another displacement ξ which compresses the next spring in the chain by ξ and so on. So for any given mass, the finial force F on the end mass will cause any mass in the chain to feel a force F.

(c) As we stated before $L_i = (N_i - 1)\ell$. Now for a given force F acting on the end mass of a chain we have showed in the previous part that any mass in the chain feels the same force F. So each mass in the chain will be displaced by an amount $\xi = \frac{F}{k}$. The total displacement is then $\Delta L = (N_i - 1)\xi$. Putting this together we have,

$$\frac{\Delta L}{L} = \frac{(N_i - 1)\xi}{(N_i - 1)\ell} = \frac{\xi}{\ell}.$$

(d) Young's modulus is defined as $Y = -\frac{FL}{A\Delta L}$. Drawing from our results in parts (a), (b), and (c) we can talk about Young's modulus the small unit cell consisting of one mass. Then,

$$Y = -\frac{F\ell}{A\xi}.$$

Recall that $\frac{F}{\xi} = -k$,

$$Y = \frac{k\ell}{A}.$$

Since we're working on the scale of a unit cell, we know $A = \ell^2$. So, $Y = \frac{k}{\ell}$.

(e) From part (b),

$$\omega_n \approx \sqrt{\frac{k\ell^2}{m}} \left(\frac{n\pi}{L}\right).$$

Substitute, $\frac{k}{\ell} = Y$ and $\frac{m}{\ell^3} = \rho$.

$$\omega_n \approx \sqrt{\frac{k\ell^3}{m\ell}} \left(\frac{n\pi}{L}\right) = \sqrt{\frac{Y}{\rho}} \left(\frac{n\pi}{L}\right).$$

(f) The units of $\sqrt{\frac{Y}{\rho}}$ are,

$$\left[\sqrt{\frac{Y}{\rho}} = \sqrt{\frac{\left(\frac{kgm^2}{m^3 s^2}\right)}{\frac{kg}{m^3}}} = \frac{m}{s}\right].$$

Similarly,

$$\left[\sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{\frac{kgm}{s^2}}{\frac{kg}{m}}} = \frac{m}{s}\right]$$

(g) The lowest frequency normal mode has $n = 1$. Recall that $T_n = \frac{2\pi}{\omega_n}$ so we have,

$$\omega_n =\approx \frac{2\pi}{T_1} = \sqrt{\frac{Y}{\rho}}\left(\frac{n\pi}{L}\right)$$

$$T_1 = 2L\sqrt{\frac{\rho}{Y}}.$$

For our piece of quartz, $L = 0.01$ m, $\rho = 2.65 \times 10^3$ kg/m^3, and $Y = 85 \times 10^9$ N/m, so $T_1 = 1.1 \times 10^{-5}$ seconds, or about 10 μ s.

(h) For our piece of jello, $L = 0.01$ m, $\rho = 1 \times 10^3$ kg/m^3, and $Y = 100$ N/m, so $T_1 = 0.06$ seconds. (You can watch jello jiggle – it's slow.)

5. Banjo Strings:

We know that the fundamental mode of a string has the following relation

$$\omega_1 = 2\pi f_1 = \frac{\pi}{L}\sqrt{\frac{\tau}{\mu}} \tag{24}$$

, $\lambda = 2L$ where L is the length of the string, τ is the tension, and μ is the mass density. Rewriting:

$$f_1 = \frac{1}{2L}\sqrt{\frac{\tau}{\mu}} \tag{25}$$

or

$$\mu = \tau/(2f_1L)^2. \tag{26}$$

(a) Plugging in the numbers, we find that the mass densities are μ = (3.01, 3.70, 6.30, 14.90, 3.01) $\times 10^{-4}$ kg/m

(b) Using the 3.01 $\times 10^{-4}$ kg/m material for all the strings, we invert the above equation to obtain the "new" tension values: $\tau = (2fL)^2\mu$ = 48.64 , 34.57 , 21.77 , 12.24 , and 48.69 Newtons.

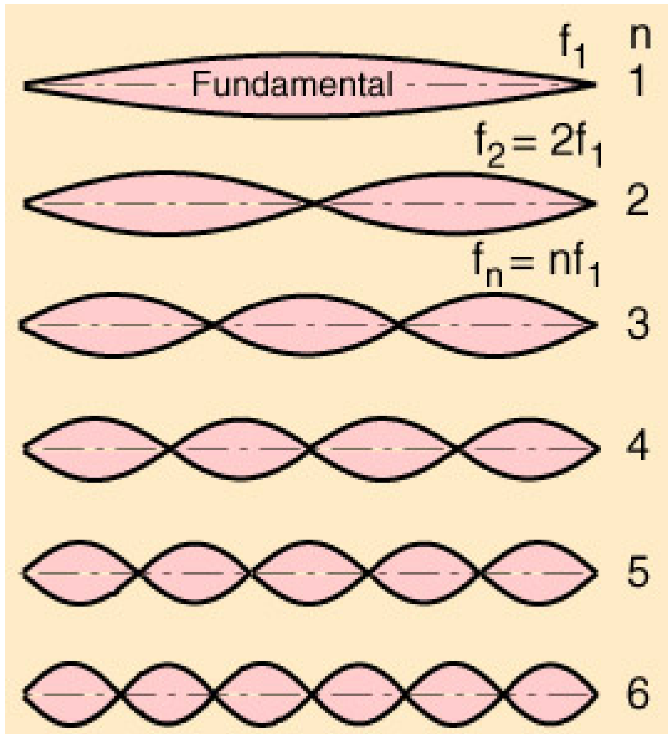

Figure 1: Various modes of a guitar string. Notice that the odd modes all have a maximum at the center of the string and the even mode all have a node at the center. Image obtained from Hyperphysics (http://hyperphysics.phy-astr.gsu.edu).

6. Plucking a String:

   When plucking a guitar string at the center we create an initial displacement that is maximal at the center of the string. All of the odd n modes have maxima at the center of the string. The even modes, on the other hand, have nodes (zeros) at the center of the string. Therefore the even modes cannot contribute to our imposed string deformation; conversely, our deformation cannot excite the even modes, since it doesn't "overlap" with their shape.