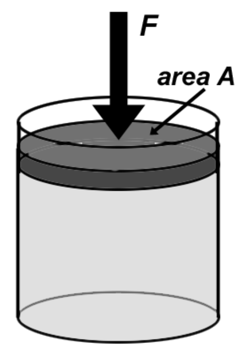
Prof. Raghuveer Parthasarathy
University of Oregon; Fall 2007

Physics 351 – Vibrations and Waves

Problem Set 8

Due date: Wednesday, Nov. 28, 5 pm. No late homework will be accepted; I intend to promptly post the solutions, so that you can study them before the Final Exam.
Reading: French Chapters 6 (skim 176-189) and 7 (to p. 223).

(1, 9 pts**) Normal modes of a gas-filled pipe.** The vibrations of a gas in a pipe (e.g. air in an organ pipe) are very similar to the longitudinal vibrations of a solid that we discussed in class and in Problem Set 7. We found that the resonant frequencies are related to the Young's modulus (Y) and the density (ρ) of the material. In parts (*a*) and (*b*) we'll determine the Young's modulus of a cylinder of **gas**, as a function of pressure.



(a, 1 pt.**)** Consider a piston – a closed cylinder of cross-sectional area A with a moveable plunger on top (see Figure). Applying a force (F) to the plunger, the pressure changes (by Δp) and the volume changes (by ΔV). Recalling that the Young's modulus is defined by

$Y = -\frac{F/A}{\Delta l / l}$, where ℓ is the cylinder length, **show that** $Y = -V\frac{\Delta p}{\Delta V}$.

(b, 3 pts.**)** For a gas under adiabatic conditions (meaning no heat flows into or out of the system through the walls), pressure and volume are related by $pV^{\gamma} = C$, where C and γ are constants. (We'll prove this next quarter.) Considering small changes (so our "Δ"s become "d"s), **show that** $Y = \gamma p$.

(c, 2 pts.**)** The value of γ depends on the structure of the gas, as we'll also see next quarter. For a monatomic ideal gas, $\gamma = 5/3$. For a diatomic gas, $\gamma = 7/5$. During a thunderstorm we recall that we can determine the distance of a lightning strike by counting the number of seconds between the lightning (which we see "immediately") and the thunder (which travels at the speed of sound) – the distance, in kilometers, is the number of seconds divided by 3. (For miles divide by 5). From Problem Set 7 we recall that the speed v that enters the wave equation (and that equals the speed of sound in the medium) is given by $v = \sqrt{Y/\rho}$. Take the density of air to be about 1.2 kg/m^3, and the pressure to be atmospheric pressure (101,000 Pa). From our "rule of thumb" for lightning strikes, would you say that air is mostly composed of monatomic or diatomic molecules? (Note that our counting statement is not very precise, so you won't get a very exact answer!)

(d, 2 pts.**)** A pipe of length L that is closed on one end and open on the other has boundary conditions equivalent to those of a rod clamped at one end and free at the other – i.e. a node for the vibrations at the closed end and free vibration at the open end. (See Figure 6-7 in the text.)

Start from the wave equation, $\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \xi}{\partial t^2}$, where ξ is the longitudinal displacement of molecules in the gas from their equilibrium position; look for normal modes (i.e. solutions of the form $\xi(x,t) = f(x)\cos(\omega t)$); and apply the boundary conditions

$$\xi(x=0)=0; \quad \text{and} \quad \left.\frac{d\xi}{dx}\right|_{x=L} = 0$$

to derive an expression for the normal mode frequencies (ω_n) in terms of L, γ, *p*, and ρ.

(e, 1 pt.**)** Calculate the frequencies *f* (=ω/2π) of the first three normal modes of a tube of length *L* = 20cm closed at one end, filled with air at standard temperature and pressure.

(2, 4 pts.**) Energies of a vibrating string, revisited.** The transverse displacement $y_n(x,t)$ of a vibrating string of length *L*, pinned at the ends, in normal mode *n* is given by

$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\omega_n t - \phi_n\right)$, where A and ϕ are the amplitude and phase set by the initial conditions, as usual. The normal mode frequencies are $\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}$. In class we evaluated the total energy by considering a moment in time when $y = 0$ – i.e. the string is crossing the "horizontal" – at which point the potential energy is zero and so the total (conserved) energy is equal to the kinetic energy. Now let's consider a different special time, the time at which the string reaches its maximum amplitude, at which $\dot{y} = 0$ and the energy is purely potential energy. Consider a small segment of the string, from *x* to *x*+d*x*.

(a, 2 pts.**)** When deriving the wave equation for the 1D string we showed that

$\tau\, dx \frac{\partial^2 y}{\partial x^2} = \mu\, dx \frac{\partial^2 y}{\partial t^2}$, where τ is the tension (force) of the string and μ is the mass density (mass per unit length). Note that the right side of the equation is simply "*ma*" (mass times acceleration) for the segment, and the left side must therefore be the net force on the segment (dF). Using the expression for $y_n(x,t)$ above, show that the force can be written as $dF = -T\, dx\left(\frac{n\pi}{L}\right)^2 y_n$, i.e. a restoring force linearly proportional to *y*, just like a simple spring. From this, derive an expression for dU, the potential energy of this segment.

(b, 2 pts.**)** Integrate dU from *x*=0 to *x*=*L* to determine the potential energy of the string, U, and therefore the total energy (at any time) of the vibrating string in mode *n*.

(3, 6 pts**) Fourier series.** Sketch the following functions, defined on the interval 0 ≤ *x* ≤ L, and find their Fourier series:

(a, 3 pts.**)** $y(x) = C\left(x^2 - xL\right)$

(b, 3 pts.**)** $y(x) = A\sin(7\pi x / L)$

You may find the following useful: $\int x^n \sin(ax)dx = -\frac{x^n}{a}\cos(ax) + \frac{n}{a}\int x^{n-1}\cos(ax)dx$ (for *n* > 0).

(4, 6 pts.**) Normal modes of a 2D rectangular sheet.** The wave equation describing vibrations of the height (*h*) of a 2-dimensional sheet is $\nabla^2 h = \frac{1}{v^2}\frac{\partial^2 h}{\partial t^2}$, which in Cartesian coordinates is simply:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 h}{\partial t^2}.$$

(a, 1 pt.**)** The velocity v is given by $v = \sqrt{S/\sigma}$, where S is the surface tension – the force holding the sheet taut divided by the length of the sheet's perimeter – and σ is the 2D mass density (mass per unit area). Show that $\sqrt{S/\sigma}$ has dimensions of velocity.

(b, 3 pts.**)** For a rectangular sheet of size $L_x \times L_y$ that is pinned at the edges, the normal modes of vibration are given by

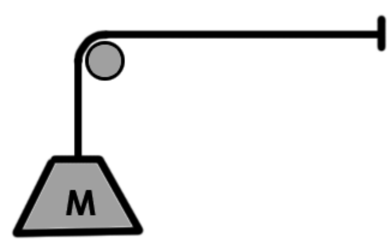
$$h(x,y,t) = C_{nm}\sin\left(\frac{n\pi x}{L_x}\right)\sin\left(\frac{m\pi x}{L_y}\right)\cos(\omega_{nm}t)$$

$$\omega_{nm} = v\sqrt{\left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2},$$

where n and m are positive integers. The spatial parts look just like those of a 1D string in the x-direction multiplied by a 1D string in the y-direction. Verify that this $h(x,y,t)$ is a solution to the wave equation. (If you are unfamiliar with partial derivatives: $\frac{\partial}{\partial x} f(x,y)$ means differentiate with respect to x, treating y as fixed.)

(c, 2 pts.**)** Consider a square sheet, so $L_x = L_y$ (call it L). How many normal modes are there (i.e. how many (m, n)) with frequencies less than 4 times ω_{11}?

(5, 4 pts.**) A traveling pulse.** A string is draped over a pulley so that the tension pulling it can be provided by a hanging mass, as shown. The mass weighs 30 times as much as a segment of string of length L. How long would it take a pulse (a wave traveling along the string) to travel a distance $L/2$, if L is 20 meters?



(6, 4 pts.**) A traveling wave.** A wave traveling along a string is described by the equation $y = 0.2\sin\left[\pi(0.2x - 30t)\right]$, where x and y are in meters and t is measured in seconds.

(a, 2 pts.**)** Find the amplitude, wavelength, wavenumber, frequency, period, and velocity of the wave.

(b, 2 pts.**)** Find the maximum transverse (i.e. along the y-direction) speed of any particle in the string.