Physics 351: Problem Set 8 Solutions

1. **Normal Modes of a Gas Filled Pipe:**

   (a) We are given Young's modulus as:

   $$Y = -\frac{F/A}{\Delta\ell/\ell}.$$

   We want to relate this quantity to the pressure and volume of the gas cylinder. We are given a way to relate force and area to pressure:

   $$\Delta p = \frac{F}{A}.$$

   Moreover, since the cross-sectional area of the cylinder never changes during the compression we can relate the change in volume of the gas to the change in piston height,

   $$\frac{\Delta\ell}{\ell} = \frac{\Delta\ell A}{\ell A} = \frac{\Delta V}{V}.$$

   Pulling this all together we have:

   $$Y = -\frac{F/A}{\Delta\ell/\ell} = -V\frac{\Delta p}{\Delta V}.$$

   (b) The $\Delta p/\Delta V$ term measures how the pressure of the cylinder changes as a function of volume change. But we know exactly what this change is since we know the equation of state for this gas filled cylinder ($p = C/V^\gamma$),

   $$\frac{\Delta p}{\Delta V} = \frac{dp}{dV} = -\gamma\frac{C}{V^{\gamma+1}} = -\gamma\frac{C}{VV^\gamma} = -\gamma\frac{p}{V}.$$

   Plugging this in to our Young's modulus gives us:

   $$Y = -V\frac{\Delta p}{\Delta V} = -V\left(-\gamma\frac{p}{V}\right) = -\gamma p.$$

   (c) From our "rule of thumb" we know that the speed of sound is $v = \frac{1}{3}$ km / s, or $v = 333$ m/s. Solving our result from (b) for γ,

   $$\gamma = Y/p = \rho v^2/p. \tag{1}$$

   Plugging in numbers, $\gamma = 1.3$, which is closer to $7/5 = 1.4$ than $5/3 = 1.67$. Therefore, it looks like air is mostly composed of diatomic molecules. (Which it is – it's overwhelminly N_2 and O_2).

(d) First, we've assumed a solution to the wave equation of the form $\xi(x,t) = f(x)\cos(\omega t)$ so let's plug it into the wave equation and see if that helps us simplify the problem of finding normal mode frequencies.

$$\begin{aligned}
\frac{\delta^2 \xi}{\delta x^2} &= \frac{1}{v^2}\frac{\delta^2 \xi}{\delta t^2} \\
\frac{\delta^2 f}{\delta x^2}\cos(\omega t) &= -\frac{\omega^2}{v^2} f(x)\cos(\omega t) \\
\frac{\delta^2 f}{\delta x^2} &= -\frac{\omega^2}{v^2} f(x).
\end{aligned}$$

We have turned our coupled differential equation with two variables into a differential equation involving only position. Moreover, we should recognize this differential equation as our familiar harmonic oscillator, which has solutions of the form $A\sin((\omega/v)x + \alpha)$. Now let's turn to the boundary condition $\xi(x = 0) = 0$.

$$\xi(x = 0) = f(0)\cos(\omega t) = 0.$$

But this condition holds true for all time so,

$$0 = f(0) = A\sin((\omega/v)(0) + \alpha) = A\sin(\alpha).$$

We ignore the trivial solution $A = 0$ and instead turn to $\sin(\alpha) = 0$. From here we now pick $\alpha = 0$. Then next boundary condition yields,

$$\frac{d\xi(x = L)}{dx} = \frac{df(x = L)}{dx}\cos(\omega t) = 0.$$

This once again is true for all time so,

$$0 = \frac{df(x = L)}{dx} = (\omega/v)A\cos(\omega_s L).$$

Since A and (ω/v) cannot be zero we know $\cos((\omega/v)L) = 0$ which gives us $(\omega/v)L =$ an odd multiple of $\pi/2$, i.e. $= (n - 1/2)\pi$ for $n = 1, 2, 3, \ldots$. Solving for ω, plugging in our above expression for v from part (b) gives us,

$$\omega = \frac{(n - \frac{1}{2})\pi v}{L} = \frac{(n - \frac{1}{2})\pi}{L}\sqrt{\frac{\gamma p}{\rho}}.$$

(e) Using our STP values for air from part (b) and $L = 20$cm we can use

$$f = \omega/(2\pi) = \frac{(n - \frac{1}{2})}{2L}\sqrt{\frac{\gamma p}{\rho}}$$

to obtain:

$$\begin{aligned}
f_1 &= 427 \text{ Hz} \\
f_2 &= 1281 \text{ Hz} \\
f_3 &= 2135 \text{ Hz}
\end{aligned}$$

2. **Energies of a Vibrating String, revisited:**

Given:

$$y_n = A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \phi_n), \quad ; \omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}. \tag{2}$$

The speed:

$$\dot{y}_n = -\omega_n A_n \sin\left(\frac{n\pi x}{L}\right)\sin(\omega_n t - \phi_n), \tag{3}$$

which is zero at t^* given by $\omega_n t^* - \phi_n = 0$ (or any integer multiple of π).

(a) As noted in the assignment, the force on the string is given by

$$dF = \tau dx \frac{\partial^2 y}{\partial x^2}, \tag{4}$$

and so

$$dF = -\tau dx A_n \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \phi_n), \tag{5}$$

which at t^* is

$$dF = -\tau dx A_n \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right) = -\tau dx \left(\frac{n\pi}{L}\right)^2 y(x, t^*) \tag{6}$$

This looks like a spring ($F = -ky$) with spring constant "k" given by

$$k = -\tau dx \left(\frac{n\pi}{L}\right)^2 \tag{7}$$

Therefore the energy associated with this piece of string is "$\frac{1}{2}ky^2$",

$$dU = -\frac{1}{2}\tau dx \left(\frac{n\pi}{L}\right)^2 [y(x, t^*)]^2 \tag{8}$$

which we can integrate over x to get the total potential energy , U.

(b) Integrating, as noted above, $U = \int dU$, using the integral of $\sin^2$ that we're familiar with:

$$\begin{aligned} U &= \int_0^L \frac{1}{2}\tau \; dx \; \left(\frac{n\pi}{L}\right)^2 A_n^2 \sin^2\left(\frac{n\pi x}{L}\right) \\ &= \frac{1}{2}\tau \left(\frac{n\pi}{L}\right)^2 A_n^2 \frac{L}{n\pi}\frac{n\pi}{2} = \frac{1}{4}\frac{\tau}{L} n^2 \pi^2 A_n^2, \end{aligned}$$

This is the potential energy at time t^* – therefore, it is the *total energy at all times.*

3. **Fourier Series:**

The general form of the Fourier Series of a function y defined over $x = 0$ to L is:

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

The coefficients that define the series are:

$$B_n = \frac{2}{L}\int_0^L y(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

(a) The parabolic function
$y = C(x^2 - xL)$. Let's evaluate B_n (making use of the given integration formula):

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L y(x)\sin(\frac{n\pi x}{L})dx && (9)\\ &= \frac{2}{L}\int_0^L Cx^2\sin(\frac{n\pi x}{L})dx. - \frac{2}{L}\int_0^L CLx\sin(\frac{n\pi x}{L})dx && (10)\end{aligned}$$

Defining $u = n\pi x/L$, so $du = n\pi dx/L$,

$$\begin{aligned} B_n &= \frac{2C}{L}\left(\frac{L}{n\pi}\right)^3\int_0^{n\pi} u^2\sin(u)du && (11)\\ &\quad -2C\left(\frac{L}{n\pi}\right)^2\int_0^{n\pi} u\sin(u)du && (12)\\ & && (13)\end{aligned}$$

$$\begin{aligned} B_n &= \frac{2C}{L}\left(\frac{L}{n\pi}\right)^3\left[-u^2\cos(u)|_0^{n\pi} + 2\int_0^{n\pi} u\cos(u)du\right] && (14)\\ &\quad -2C\left(\frac{L}{n\pi}\right)^2\left[-u\cos(u)|_0^{n\pi} + \int_0^{n\pi}\cos(u)du\right] && (15)\\ & && (16)\end{aligned}$$

$$\begin{aligned} B_n &= \frac{2C}{L}\left(\frac{L}{n\pi}\right)^3 2\int_0^{n\pi} u\cos(u)du && (17)\\ &\quad -2C\left(\frac{L}{n\pi}\right)^2[-n\pi\cos(n\pi) + \sin(n\pi) - 0] && (18)\\ &\quad +\frac{2C}{L}\left(\frac{L}{n\pi}\right)^3[-(n\pi)^2\cos(n\pi)] && (19)\end{aligned}$$

$$\text{Note that } \sin(n\pi) = 0 \qquad (20)$$

$$(21)$$

$$
\begin{aligned}
B_n &= \frac{4CL^2}{(n\pi)^3}\int_0^{n\pi} u\cos(u)du & (22)\\
&+\frac{2CL^2}{n\pi}\cos(n\pi) - \frac{2CL^2}{n\pi}\cos(n\pi) & (23)\\
\text{Integrate by parts...} & & (24)\\
B_n &= -\frac{4CL^2}{(n\pi)^3}(1-\cos(n\pi)). & (25)
\end{aligned}
$$

Thinking about cosines, we see that:

$$
\begin{aligned}
B_n &= -\frac{8CL^2}{(n\pi)^3} \text{ for odd } n & (26)\\
&= 0 \text{ for even } n. & (27)
\end{aligned}
$$

(By plotting $y(x)\sin(n\pi x/L)$, we could have realized ahead of time that all the even B_n would integrate to zero.)

(b) The function is $y = A\sin(\frac{7\pi x}{L})$. Thus

$$B_n = \frac{2}{L}\int_0^L A\sin(\frac{7\pi x}{L})\sin(\frac{n\pi x}{L})dx$$

From the orthogonality of sine functions (recall the lecture on Fourier Series),

$$
\begin{aligned}
\int_0^L \sin(\frac{m\pi x}{L})\sin(\frac{n\pi x}{L})dx = &\quad L/2, \text{ if integer } m=n & (28)\\
&\quad 0, \text{ otherwise,} & (29)
\end{aligned}
$$

we know that all the B_n are zero except B_7, which (think about the Fourier Series itself) is simply B_7 = A. (You can do the integral, if you want.) This function is its own Fourier series expansion!

4. **2D Sheet:**

(a) Unit analysis:

$$\left[\sqrt{\frac{S}{\sigma}} = \sqrt{\frac{N/m}{kg/m^2}} = \sqrt{\frac{kg/s^2}{kg/m^2}} = \sqrt{\frac{m^2}{s^2}} = \frac{m}{s}\right]$$

(b) To test if our given $h(x, y, t)$ is a solution we need only plug it into the differential equation and show that our expression for the normal mode frequencies matches the given expression. Let's get the derivatives out of the way first,

$$\begin{aligned}
\frac{\delta^2 h}{\delta x^2} &= -C_{nm}\left(\frac{n\pi}{L_x}\right)^2 \sin\left(\frac{n\pi}{L_x}x\right)\sin\left(\frac{m\pi}{L_y}y\right)\cos(\omega_{nm}t) = -\left(\frac{n\pi}{L_x}\right)^2 h(x,y,t) \\
\frac{\delta^2 h}{\delta y^2} &= -C_{nm}\left(\frac{m\pi}{L_y}\right)^2 \sin\left(\frac{n\pi}{L_x}x\right)\sin\left(\frac{m\pi}{L_y}y\right)\cos(\omega_{nm}t) = -\left(\frac{m\pi}{L_y}\right)^2 h(x,y,t) \\
\frac{\delta^2 h}{\delta t^2} &= -\omega_{nm}^2 \sin\left(\frac{n\pi}{L_x}x\right)\sin\left(\frac{m\pi}{L_y}y\right)\cos(\omega_{nm}t) = -\omega_{nm}^2 h(x,y,t).
\end{aligned}$$

Plugging these derivatives into the wave equation gives us,

$$\begin{aligned}
-\frac{1}{v^2}\omega_{nm}^2 h(x,y,t) &= -\left(\frac{n\pi}{L_x}\right)^2 h(x,y,t) + -\left(\frac{m\pi}{L_y}\right)^2 h(x,y,t) \\
\frac{\omega_{nm}^2}{v^2} &= \left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 \\
\frac{\omega_{nm}^2}{v^2} &= \left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 \\
\omega_{nm} &= v\sqrt{\left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2}.
\end{aligned}$$

(c) With $L_x = L_y = L$,

$$\omega_{nm} = \frac{v\pi}{L}\sqrt{n^2 + m^2}. \tag{30}$$

and so

$$\omega_{11} = \frac{v\pi}{L}\sqrt{2}. \tag{31}$$

Considering the next few modes:

$$\begin{aligned}
\omega_{12} &= \omega_{21} = \frac{v\pi}{L}\sqrt{5} = \sqrt{5/2}\omega_{11} && (32)\\
\omega_{13} &= \omega_{31} = \frac{v\pi}{L}\sqrt{10} = \sqrt{5}\omega_{11} && (33)\\
\omega_{14} &= \omega_{41} = \frac{v\pi}{L}\sqrt{17} = \sqrt{17/2}\omega_{11} && (34)\\
\omega_{15} &= \omega_{51} = \frac{v\pi}{L}\sqrt{26} = \sqrt{25/2}\omega_{11} && (35)\\
\omega_{16} &= \omega_{61} = \frac{v\pi}{L}\sqrt{37} = \sqrt{37/2}\omega_{11} > 4\omega_{11} && (36)\\
\omega_{22} &= \frac{v\pi}{L}\sqrt{8} = 2\omega_{11} && (37)\\
\omega_{23} &= \omega_{32} = \frac{v\pi}{L}\sqrt{13} = \sqrt{13/2}\omega_{11} && (38)\\
\omega_{24} &= \omega_{42} = \frac{v\pi}{L}\sqrt{20} = \sqrt{10}\omega_{11} && (39)\\
\omega_{25} &= \omega_{52} = \frac{v\pi}{L}\sqrt{29} = \sqrt{29/2}\omega_{11} && (40)\\
\omega_{26} &= \omega_{62} = \frac{v\pi}{L}\sqrt{40} = \sqrt{20}\omega_{11} > 4\omega_{11} && (41)\\
\omega_{33} &= \frac{v\pi}{L}\sqrt{18} = 3\omega_{11} && (42)\\
\omega_{34} &= \omega_{43} = \frac{v\pi}{L}\sqrt{25} = 5\omega_{11} > 4\omega_{11} && (43)\\
\omega_{44} &= \frac{v\pi}{L}\sqrt{32} = 4\omega_{11} && (44)\\
& && (45)
\end{aligned}$$

So we see that there are **17 modes** with frequencies less than $4\omega_{11}$ – these are modes (m, n) = (1,1), (1,2), (2,1), (1,3), (3,1), (1,4), (4,1), (1,5), (5,1), (2,2), (2,3), (3,2), (2,4), (4,2), (2,5), (5,2), and (3,2). Note that we can't count e.g. (2,2) as two modes – flipping m and n leads to a mode whose spatial structure looks the same.

5. A Traveling pulse

First let's recall that,

$$v = \sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{\tau L}{M}}$$

for a string, expressing μ in terms of the mass of a segment of length L. The tension is created by a weight that has a mass 30 times that of the string. In other words, $\tau = 30Mg$. Plugging this into the expression above gives us,

$$v = \sqrt{30gL}.$$

The pulse takes time t to travel a distance $L/2$, and hence has speed $v = L/2t$. Therefore

$$\begin{aligned} v &= \sqrt{30gL} = \frac{L}{2t} \\ t &= \frac{1}{2}\sqrt{\frac{L}{30g}} \end{aligned}$$

Using $L = 20$ m and $g = 9.8$ m/s^2, $t = 0.13$ seconds.

6. **A traveling wave.**

(a). We're given: $y(x,t) = 0.2\sin\left[\pi(0.2x - 30t)\right]$ meters (with t in seconds). We know from French equation 7-6 that propagating waves take the form $y = A\sin(\frac{2\pi}{\lambda}(x - vt))$. All we need to do is pick off the values from our given equation. The amplitude stands out first of all $A = 0.2$m. Next we can find the wavelength by noting $\frac{2\pi}{\lambda} = 0.2\pi$. This gives us $\lambda = 10$ m. We can now go on to calculate the wave number $k = \frac{2\pi}{\lambda} = \frac{\pi}{5}\mathrm{m}^{-1}$. Next we'll pick off the speed by noting $\frac{2\pi}{\lambda}v = 30\pi$. This gives us $v = 150$ m/s and then the frequency $f = \frac{v}{\lambda} = 15$ Hz. The period is $T = \frac{1}{f} = 0.067$s.

(b). The transverse speed is simply $\partial y/\partial t$ – the rate of change of y.

$$\frac{\partial y}{\partial t} = 0.04\pi\cos\left[\pi(0.2x - 30t)\right] \tag{46}$$

whose maximum is simply 0.04π meters / sec.