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Physics 353 -- Spring 2008

LECTURE: Bose Gases
& Bose-Einstein Condensation

Degenerate, non-interacting Bosons.

We'll see: Below a "critical temperature," a substantial fraction of the particles will occupy the lowest energy orbital. Other orbitals will have negligible occupancy.

"Bose-Einstein Condensation" Bose, Einstein 1920's (theory)
Onnes (Liquid helium superfluidity)
Wieman & others, true BEC, 1995.

Surprising? Consider ^4He (2p, 2n, 2e) in a $(1\,cm)^3$ cube.

Spacing in energy between $n_1, n_2, n_3 = 1,1,1$ & $2,1,1$ levels is $\approx 10^{-37}$ J $\Rightarrow$ "naively" expect large ground state occupancy at $T < 10^{-37}\,\text{J}/k_B \approx 10^{-14}$ K.

Actually, critical temperature $T_{BE} \approx 4$ K!

Interesting...

Recall $f_{BE} = \dfrac{1}{\exp\left(\frac{\varepsilon-\mu}{\tau}\right)-1}$ = occupancy of orbital of energy ε.

Be able to derive from $\mathcal{Z}$

$$= \frac{1}{\lambda^{-1}e^{\varepsilon/\tau}-1}, \text{ using } \lambda \equiv e^{\mu/\tau}.$$

Note that we must have $f_{BE} \geq 0$ (can't have negative # of particles!)

$\Rightarrow \lambda^{-1}e^{\varepsilon/\tau}-1 \geq 0$ ($\frac{1}{\#}$ also ≥ 0) $\Rightarrow e^{(\varepsilon-\mu)/\tau} \geq 1$

$\Rightarrow \frac{\varepsilon-\mu}{\tau} \geq 0 \Rightarrow \underline{\mu \leq \varepsilon}$ for any orbital.

$\Rightarrow \boxed{\mu \leq \varepsilon_0} \leftarrow$ lowest ε (ground state)

Bosons: chem. pot. can't be greater than the energy of the lowest orbital.

If we define $\varepsilon_0 = 0$, then $\mu \leq 0$, $\lambda \leq 1$.

Let's call the avg. # particles in the ground state N_0.

$$N_0 = f_{BE}(\varepsilon_0) = \frac{1}{\exp\left(\frac{\varepsilon_0-\mu}{\tau}\right)-1}.$$

If τ is sufficiently small, N_0 must be large physically, (particles $\to$ ground)

So denominator must be small, so $\exp\left(\frac{\varepsilon_0-\mu}{\tau}\right)$ is near 1,

so $\frac{\varepsilon_0-\mu}{\tau}$ is very small (& positive, as above).

$\Rightarrow$ Taylor series $\exp\left(\frac{\varepsilon_0-\mu}{\tau}\right)-1 \approx 1 + \frac{\varepsilon_0-\mu}{\tau} + \dots - 1$

$\Rightarrow$ $N_0 \approx \frac{\tau}{\varepsilon_0-\mu}$ when $N_0 >> 1$

As $\tau \to 0$, $\mu \to \varepsilon_0$ (so N_0 is finite)

For τ slightly >0, μ slightly $< \varepsilon_0$

How large can τ be to maintain $N_0 >> 1$?

What determines μ? $\sum_{\text{orbitals}, i} \underbrace{\frac{1}{e^{(\varepsilon_i-\mu)/\tau}-1}}_{f(\varepsilon_i)} = \underbrace{N}_{\text{total \# particles}}$.

As in the FD case, let's try to convert to an integral:

$$N = \int_0^\infty \underbrace{D(\varepsilon)}_{\text{dens. of states}} \underbrace{\frac{1}{e^{(\varepsilon-\mu)/\tau}-1}}_{f_{BE}} d\varepsilon$$

valid if $\tau >> \varepsilon_0$, so a "continuum" of occupied levels. (Red Flag!)

Consider 3D, spin 0 bosons. $2s+1 = 1 \Rightarrow$ only 1 spin state

same particle-in-a-box as our 3D fermions

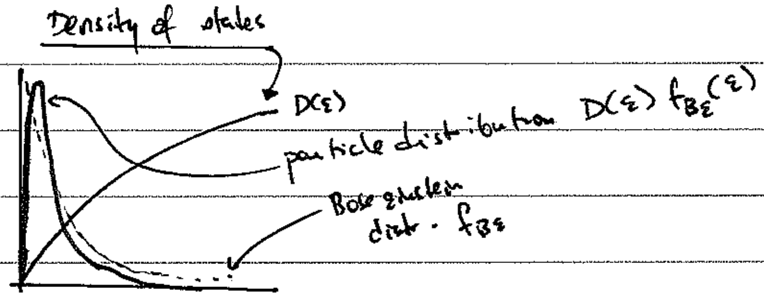
$\Rightarrow$ $D(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}$

just $\div 2$ of our electron $D(\varepsilon)$.

In general: $\langle A \rangle = \int A\, f(\varepsilon)\, D(\varepsilon)\, d\varepsilon$

dens. of states – a property of the "box"

distribution – FD or BE or Boltzmann



(Eq. 1) $$N = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_{0 \text{ or } \varepsilon_0}^{\infty} \frac{\varepsilon^{1/2}}{e^{(\varepsilon-\mu)/\tau}-1}\, d\varepsilon .$$ Can't do analytically. (in general).

Recall we care about N_0 being large, and so μ near ε_0. Call $\varepsilon_0 = 0$, as we did in the limits of integral

Let's look at $\underline{\mu = 0}$. (max possible μ).

$$N = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^{\infty} \frac{\sqrt{\varepsilon}\, d\varepsilon}{e^{\varepsilon/\tau}-1}$$

$$N = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\tau\right)^{3/2} \underbrace{\int_0^{\infty} \frac{\sqrt{x}\, dx}{e^x-1}}_{\text{dimensionless integral, } = 1.306\sqrt{\pi}}$$ with $x = \varepsilon/\tau$ (see K&K)

Skip ~~Recall~~ $\cancel{n_Q = \left(\frac{m\tau}{2\pi\hbar^2}\right)^{3/2} \leftarrow \text{depends on } \tau}$

Combining numbers

(Eq. 2) $$N = (2.612)\left(\frac{m\tau}{2\pi\hbar^2}\right)^{3/2} V$$

This is nonsensical : N depends on τ ??

There's some particular τ at which the above relation is true ; call it τ_c : $N = 2.612\left(\frac{m\tau_c}{2\pi\hbar^2}\right)^{3/2} V$

Suppose $\tau > \tau_c$. Eq. 2 suggests $N >$ true N.

But return to Eq. 1 : As τ rises, μ gets further <u>below</u> 0 or ε_0, as we saw earlier.

So $e^{(\varepsilon-\mu)/\tau}$ is larger, so the integral in Eq. 1 is <u>smaller</u>, in such a way as to keep the right side of Eq. 1 $= N$.

(as usual can use this to determine μ)

Suppose $\tau < \tau_c$. More interesting.

Nothing about the behavior of μ helps us, since $\mu = 0$ (ε₀) is as large as μ can get.

What's going on?

our $N = \sum_i \frac{1}{e^{(\varepsilon_i - \mu)/\tau} - 1} \longrightarrow \int_0^\infty D(\varepsilon) \underbrace{\frac{1}{e^{(\varepsilon-\mu)/\tau} - 1}}_{f_{BE}} d\varepsilon$

is invalid!

Why? As $\varepsilon \to 0$ ($= \varepsilon_0$), $D(\varepsilon) \to 0$ (as $\sqrt{\varepsilon}$) and $f_{BE}(\varepsilon) \to \infty$

(since $\mu \approx 0$ (ε_0) at low τ).

The sum in $N = \sum_i \cdots$ has an "infinite spike" at $\varepsilon = 0$ that is not correctly represented by the integral.

The integral should correctly represent the particles in all the states away from the ground orbital (where there's no spike).

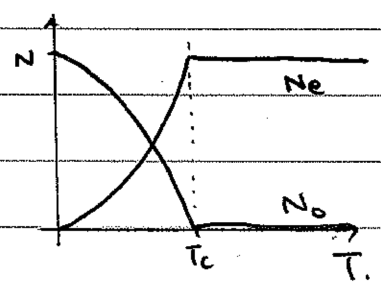
So really our $N = \int D(\varepsilon) f(\varepsilon)\, d\varepsilon$ is N_e, the number of particles in the excited states.

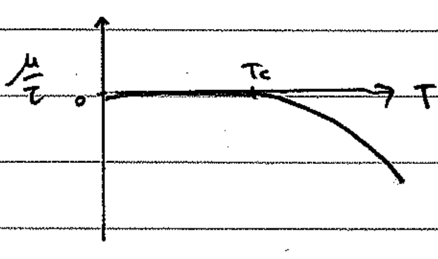
So $N_e = (2.612) \left(\frac{m\tau}{2\pi\hbar^2} \right)^{3/2} V$ for $\tau < \tau_c$

or, using the def. of τ_c, $N_e = \left(\frac{\tau}{\tau_c} \right)^{3/2} N$ (for $\tau < \tau_c$)

The rest of the atoms must be in the ground state,

so $N_0 = N - N_e = \left[1 - (\tau/\tau_c)^{3/2} \right] N$ ($\tau < \tau_c$).





Abrupt accumulation of bosons in the ground state at $\tau < \tau_c$: BOSE – EINSTEIN CONDENSATION

τ_c = "condensation temperature".

ground state atoms = "condensate".

note $N = 2.612 \left(\frac{m\tau_c}{2\pi\hbar^2}\right)^{3/2} V$; recall $n_Q \equiv \left(\frac{m\tau}{2\pi\hbar^2}\right)^{3/2}$

$\Rightarrow \frac{N}{V} = 2.612\, n_Q(\tau_c)$; i.e. at τ_c, the concentration $\frac{N}{V} \approx$ the quantum concentration n_Q

meaning: when "wavefunctions" overlap, quantum properties can be dramatic!

Experiments.

In 1995 – about 70 years after Bose-Einstein –

the first Bose-Einstein condensate of weakly interacting Bosons: Rubidium-87, laser cooling + trapping. U Colorado. Wieman et al.

$V \approx 10^{-15}\,m^3$; $\tau_c \approx 10^{-7}$ K.

Nobel Prize.

Since then, dilute gases of sodium, hydrogen, etc.

"New state" of matter — all particles in the same orbital.

Phenomena like coherence — diffraction of matter waves. Interferometry. "Quantum" effects.

BEC (sort of) for strongly INTERACTING BOSONS

- Superfluidity in helium-4. Zero viscosity phase below 2.17 K.
- Superfluidity in helium-3. ?! A fermion. But $^3He - {^3He}$ pair is a boson; pairs form, condense. Superfluid below $\approx$ 3mK
- Superconductivity. Again, Fermion pairs.

note: Presentations