Physics 353: MIDTERM EXAMINATION

Format: Closed book. No calculators. There are 4 problems.
Total Time: 50 minutes.
Write clearly!

POSSIBLY USEFUL RELATIONS:

NOTATION: \( P \) = pressure; \( \tau \) = temperature (absolute, “fundamental” temperature); \( T \) = conventional temperature; \( \beta \) = inverse temperature = \( 1/\tau \); \( E \) = energy; \( U \) = mean energy; \( \sigma \) = entropy; \( V \) = volume; \( F \) = Helmholtz Free Energy; \( Z \) = partition function; \( N \) = number of particles, \( n = N/V \), chemical potential \( \mu \), \( Z \) = grand partition function.

- Stirling’s approximation for factorials:
  \[ \ln(N!) \approx N \ln(N) - N. \]
- Geometric series:
  \[ \sum_{m=0}^{\infty} r^m = \frac{1}{1-r} \]
- \( F = -\tau \ln(Z) \)
- Energy of a Monatomic Ideal Gas: \( U = \frac{3}{2} N \tau \)
- Entropy of a Monatomic Ideal Gas:
  \[ \sigma = N \left( \frac{5}{2} + \ln \left( \frac{n q V}{N} \right) \right) \]
- The quantum concentration for a particle of mass \( m \): \( n_q = \left( \frac{m \tau}{2 \pi \hbar^2} \right)^{3/2} \)
- Chemical potential of an Ideal Gas:
  \[ \mu = \tau \ln \left( \frac{n}{n_q Z_{int}} \right) \]
- The activity \( \lambda \equiv e^{\mu/\tau} \)
- A relation concerning the chemical potential:
  \[ \mu = \frac{\partial F}{\partial N}_{r,\tau} \]
- \( \langle N \rangle = \frac{1}{Z} \lambda \left( \frac{\partial}{\partial \lambda} Z \right) \)
Waste heat. A heat engine operates between a hot reservoir at temperature $T_H$ and a cold reservoir at $T_L$, with the usual goal of converting the input heat $Q_H$ into as much work as possible. Define $y \equiv Q_L / Q_H$ as the fraction of heat wasted; $Q_L$ is the heat leaving the engine.

(a, 3 pts.) Derive an expression for the minimum possible $y$ as a function of $T_H$ and $T_L$.

Deriving it is important – you’ll get zero points for simply remembering the Carnot efficiency and translating it into a relation for $y$.

(b, 1 pt.) Living organisms make use of a remarkable variety of very efficient schemes for turning various forms of energy into work. As far as anyone knows, none make use of heat engines. Comment on why this might be. You may wish to note that water boils at 100°C (373 K), and might find very rough numerical calculations useful.
2, 2 pts. **Phonon heat capacity.** For *photons*, we showed that the internal energy $U \propto \tau^4$. ("$\propto$" means "proportional to.") For phonons, should we find $U \propto \tau^4$ in the low or high temperature regime? **Explain.** You need not perform any calculations.

**Answer:** Low temperature regime. The behavior of phonons is almost identical to that of photons, with one key difference being that for phonons, there is a maximum possible frequency, $\omega_{\text{max}}$, and therefore a corresponding maximum photon energy, $\hbar \omega_{\text{max}}$. If $\tau \ll \hbar \omega_{\text{max}}$, states with $\omega$ near $\omega_{\text{max}}$ will barely be occupied, and so $\omega_{\text{max}}$ may as well be infinity! So in the low temperature regime, we'd expect $U \propto \tau^4$ like the photons.

3, 6 pts. **The greenhouse revisited.** Consider the Earth to be in radiative equilibrium at temperature $T_E$, receiving and re-radiating energy from the sun. The sun has temperature $T_\odot$, radius $R_\odot$, and is a distance $D_{\odot E}$ from the Earth. Consider the Earth to have an atmosphere that is transparent to all the sun's radiation, and that absorbs a fraction $x$ of the Earth's radiation, being transparent to the remaining fraction $1-x$. The separation between the atmosphere and the Earth's surface is negligible. Derive an expression for $x$ in terms of $T_E$, $T_\odot$, etc. – i.e. an expression that allows us to characterize the greenhouse in terms of the terrestrial temperature we observe.¹

**If and only if** you are unable to do this problem: (3 pts.) Consider $x=1$ and derive an expression for $T_E$ as a function of the relevant parameters.

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¹ Incidentally, we are ignoring in all these problems the reflectivity of the Earth, which is an important factor. As you can tell, the Earth isn’t black!
4, 9 pts. **Droplet coalescence.** Consider a chemical equilibrium between a droplet of radius $R_1$ and two droplets of radius $R_2$ – i.e. the large droplet can split into two smaller ones, or two smaller ones can coalesce into one large droplet. See the illustration below. There is a potential energy $4\pi R^2 \gamma$ associated with each drop, where $\gamma$ is called the surface tension. (This situation is a model of such things as oil droplets in water, the formation of certain nanocrystals, and the cohesion of atomic nuclei. We’re ignoring here the fact that a larger drop may, of course, split into many small drops of many different sizes.) Consider droplets of radius $R_1$ as a species that behaves as an ideal solution with concentration $n_1$; denote the quantum concentration $n_{Q,1}$ and the internal partition function $Z_1$; define symbols for the droplets of radius $R_2$ similarly. (Note that the chemical potential of an ideal gas is given on Page 1.) The system is at temperature $\tau$.

(a, 4 pts.) Derive an expression for $n_1$ in terms of $n_2$, $\gamma$, $\tau$, $R_1$, $R_2$, the $n_{Q}$’s and the $Z$’s.

\[ \frac{n_1}{n_{Q_1} Z_{1n_1}} = \left( \frac{n_2}{n_{Q_2} Z_{2n_2}} \right)^2 \exp \left( \frac{4\pi \gamma}{\tau} \left( 2R_2^2 - R_1^2 \right) \right) \]
(b, 1 pt.) The above reaction must conserve volume – e.g. the volume of oil in the large droplet must equal the volume of oil in the two smaller droplets. Using this, remove $R_2$ from your expression from part (a).

\[
\begin{align*}
\text{Volume conservation:} & \quad \frac{4}{3} \pi R_1^3 = 2 \cdot \frac{4}{3} \pi R_2^3 \\
\Rightarrow & \quad R_1^3 = 2 R_2^3 \\
\Rightarrow & \quad \frac{R_1}{R_2} = 2^{\frac{1}{3}} \\
& \quad \Rightarrow \quad \frac{n_1}{n_0 \exp \left[ \frac{4 \pi R_1^2}{\gamma} \left( 2^{\frac{1}{3}} - 1 \right) \right]} \\
& \quad \text{Note:} \quad \frac{2^{\frac{1}{3}}}{R_2^2} = 2^{\frac{1}{3}} \\& = 2^{\frac{1}{3}}
\end{align*}
\]

(c, 2 pts.) Suppose $n_1$ is kept roughly constant, and the surface tension is nonzero and positive. If you want to decrease the concentration of small droplets, should you increase or decrease the surface tension? Does this result make sense? Explain.

**Answer:** From above, if $n_1$ is constant and $n_2$ is to decrease, $\exp(\gamma)$ must increase; therefore we should increase the surface tension. This makes sense: If the surface tension is higher, the energetic cost of creating two smaller droplets (whose combined surface area is larger than that of one large drop, as you can verify) is greater, shifting the equilibrium away from small droplets.

(d, 2 pts.) Suppose the surface tension is zero. Assume that $Z_1$ and $Z_2$ are roughly similar in magnitude. (This might not be a good assumption in “real life.”) Given that these are ideal solutions, which of the following statements, if any, is true: $n_2 \gg n_1$ or $n_1 \gg n_2$. Does this result make sense? Explain.

\[
\begin{align*}
\text{If } & \gamma = 0, \quad \frac{n_1}{n_0 \exp \left[ \frac{4 \pi R_1^2}{\gamma} \left( 2^{\frac{1}{3}} - 1 \right) \right]} \\
& \frac{R_1}{R_2} > 1 \text{ and } 2 R_1 > 1 \text{ so } \frac{n_1}{n_0 \exp \left[ \frac{4 \pi R_1^2}{\gamma} \left( 2^{\frac{1}{3}} - 1 \right) \right]} > 1. \\
& \text{Similarly, } \frac{n_0}{n_0 \exp \left[ \frac{4 \pi R_2^2}{\gamma} \left( 2^{\frac{1}{3}} - 1 \right) \right]} \ll 1. \\
& \text{Both are small positive numbers.} \\
& \text{The square of a small positive number is an even smaller positive number (try e.g. 0.001), so:} \\
& \frac{n_2}{n_0 \exp \left[ \frac{4 \pi R_2^2}{\gamma} \left( 2^{\frac{1}{3}} - 1 \right) \right]} > \frac{n_1}{n_0 \exp \left[ \frac{4 \pi R_1^2}{\gamma} \left( 2^{\frac{1}{3}} - 1 \right) \right]}. \\
& \text{If we imagine } R_1 \gg 1 \text{ and } R_2 \gg 1, \text{ we might be closer in magnitude,} \\
& \frac{n_2}{n_1} > 1. \\
& \text{This is as expected – in the absence of surface tension, creating lots of small droplets maximizes the entropy and lowers the free energy.}
\end{align*}
\]