

Physics 353: MIDTERM EXAMINATION

Format: Closed book. No calculators. There are 4 problems.

Total Time: 50 minutes.

Write clearly!

POSSIBLY USEFUL RELATIONS:

NOTATION: P = pressure; τ = temperature (absolute, “fundamental” temperature); T = conventional temperature; β = inverse temperature = $1/\tau$; E = energy; U = mean energy; σ = entropy; V = volume; F = Helmholtz Free Energy; Z = partition function; N = number of particles, $n = N/V$, chemical potential μ , \mathbb{Z} = grand partition function.

- Stirling’s approximation for factorials:
 $\ln(N!) \approx N \ln(N) - N$.
- Geometric series: $\sum_{m=0}^{\infty} r^m = \frac{1}{1-r}$
- $F = -\tau \ln(Z)$
- Energy of a Monatomic Ideal Gas: $U = \frac{3}{2} N\tau$
- Entropy of a Monatomic Ideal Gas:
$$\sigma = N \left(\frac{5}{2} + \ln \left(\frac{n_Q V}{N} \right) \right)$$
- The quantum concentration for a particle of mass m : $n_Q = \left(\frac{m\tau}{2\pi\hbar^2} \right)^{3/2}$
- Chemical potential of an Ideal Gas:
$$\mu = \tau \ln \left(\frac{n}{n_Q Z_{\text{int}}} \right)$$
- The activity $\lambda \equiv e^{\mu/\tau}$
- A relation concerning the chemical potential: $\mu = \left. \frac{\partial F}{\partial N} \right|_{\tau, V}$
- $\langle N \rangle = \frac{1}{\mathbb{Z}} \lambda \left(\frac{\partial}{\partial \lambda} \mathbb{Z} \right)$

(1, 4 pts.) **Waste heat.** A heat engine operates between a hot reservoir at temperature τ_H and a cold reservoir at τ_L , with the usual goal of converting the input heat Q_H into as much work as possible. Define $y \equiv Q_L / Q_H$ as the fraction of heat wasted; Q_L is the heat leaving the engine.

(a, 3 pts.) **Derive** an expression for the minimum possible y as a function of τ_H and τ_L . Deriving it is important – you'll get zero points for simply remembering the Carnot efficiency and translating it into a relation for y .

a. Energy Conservation / Thermodynamic Identity

$$\Rightarrow W + Q_L = Q_H$$

Entropy "out" $\Delta S = \frac{Q_L}{\tau_L} \geq$ Entropy "in" $\Delta S = \frac{Q_H}{\tau_H}$

$$y \equiv \frac{Q_L}{Q_H} = \text{waste heat fraction} \quad \Rightarrow \frac{Q_L}{Q_H} \geq \frac{\tau_L}{\tau_H}$$

$$\Rightarrow y = \frac{Q_L}{Q_H} \geq \frac{\tau_L}{\tau_H} \quad \boxed{y \geq \frac{\tau_L}{\tau_H}}$$

(b, 1 pt.) Living organisms make use of a remarkable variety of very efficient schemes for turning various forms of energy into work. As far as anyone knows, **none** make use of heat engines. Comment on why this might be. You may wish to note that water boils at 100 °C (373 K), and might find very rough numerical calculations useful.

b. At best, the "reservoir temperatures" used by an organism are constrained by the freezing & boiling temperatures of water. So at best, $y = \frac{0^\circ\text{C}}{100^\circ\text{C}} = \frac{273\text{K}}{373\text{K}}$
 $y \approx 3/4$ – at least this much energy must be wasted. It's hard to imagine an organism that spans ice & boiling water, so y is undoubtedly much higher! For these "biological" temperatures, a heat engine is very inefficient.

2, 2 pts. **Phonon heat capacity.** For *photons*, we showed that the internal energy $U \propto \tau^4$. (“ \propto ” means “proportional to.”) For phonons, should we find $U \propto \tau^4$ in the low or high temperature regime? **Explain.** You need not perform any calculations.

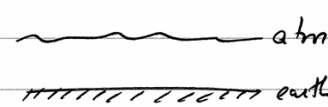
Answer: Low temperature regime. The behavior of phonons is almost identical to that of photons, with one key difference being that for phonons, there is a maximum possible frequency, ω_{max} , and therefore a corresponding maximum photon energy, $\hbar\omega_{max}$. If $\tau \ll \hbar\omega_{max}$, states with ω near ω_{max} will barely be occupied, and so ω_{max} may as well be infinity! So in the **low temperature regime**, we'd expect $U \propto \tau^4$ like the photons.

3, 6 pts. **The greenhouse revisited.** Consider the Earth to be in radiative equilibrium at temperature T_E , receiving and re-radiating energy from the sun. The sun has temperature T_\odot , radius R_\odot , and is a distance D_{SE} from the Earth. Consider the Earth to have an atmosphere that is transparent to **all** the sun's radiation, and that absorbs a **fraction** x of the Earth's radiation, being transparent to the remaining fraction $1-x$. The separation between the atmosphere and the Earth's surface is negligible. Derive an expression for x in terms of T_E , T_\odot , etc. – i.e. an expression that allows us to characterize the greenhouse in terms of the terrestrial temperature we observe.¹

If and only if you are unable to do this problem: (3 pts.) Consider $x=1$ and derive an expression for T_E as a function of the relevant parameters.

at

Solar power received earth = $P_\odot = \underbrace{\sigma_B T_\odot^4}_{\text{solar flux}} \underbrace{4\pi R_\odot^2}_{\text{fraction received by earth}} \frac{\pi R_E^2}{4\pi D_{SE}^2} = \sigma_B T_\odot^4 \pi \frac{R_\odot^2 R_E^2}{D_{SE}^2}$



atm: transparent to sun,
absorbs x from earth.

Earth's radiative Equilibrium: $P_\odot + \frac{1}{2} P_{atm} = P_E$ ← rad. by earth
 τ downward fraction from atm

Atmosphere's radiative equil: $x P_E = P_{atm}$

$P_E = \sigma_B T_E^4 4\pi R_E^2$

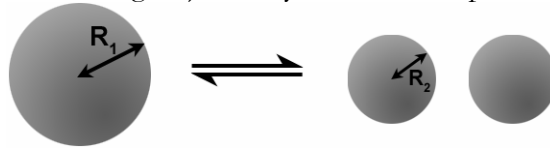
Substitute to eliminate $P_{atm} \Rightarrow P_\odot + \frac{x}{2} P_E = P_E$

$\Rightarrow \cancel{\sigma_B} T_\odot^4 \pi \frac{R_\odot^2 R_E^2}{D_{SE}^2} = \cancel{\sigma_B} T_E^4 4\pi R_E^2 (1 - \frac{x}{2}) \Rightarrow 1 - \frac{x}{2} = \left(\frac{T_\odot}{T_E}\right)^4 \frac{1}{4} \left(\frac{R_\odot}{D_{SE}}\right)^2$

Therefore $x = 2 \left[1 - \left(\frac{T_\odot}{T_E}\right)^4 \frac{1}{4} \left(\frac{R_\odot}{D_{SE}}\right)^2 \right]$

¹ Incidentally, we are ignoring in all these problems the reflectivity of the Earth, which is an important factor. As you can tell, the Earth isn't black!

4, 9 pts. **Droplet coalescence.** Consider a chemical equilibrium between a droplet of radius R_1 and two droplets of radius R_2 – i.e. the large droplet can split into two smaller ones, or two smaller ones can coalesce into one large droplet. See the illustration below. There is a potential energy $4\pi R^2\gamma$ associated with each drop, where γ is called the surface tension. (This situation is a model of such things as oil droplets in water, the formation of certain nanocrystals, and the cohesion of atomic nuclei. We're ignoring here the fact that a larger drop may, of course, split into many small drops of many different sizes.) Consider droplets of radius R_1 as a species that behaves as an ideal solution with concentration n_1 ; denote the quantum concentration $n_{Q,1}$ and the internal partition function Z_1 ; define symbols for the droplets of radius R_2 similarly. (Note that the chemical potential of an ideal gas is given on Page 1.) The system is at temperature τ .



(a, 4 pts.) Derive an expression for n_1 in terms of n_2 , γ , τ , R_1 , R_2 , the n_Q 's and the Z 's.

(a) Chem. Equilibrium: $\mu_1 - 2\mu_2 = 0$

$\mu = \mu_{\text{ideal gas}} + \mu_{\text{ext}}$ ← surface tension

$\Rightarrow \tau \ln \frac{n_1}{n_{Q,1} Z_{int,1}} + 4\pi R_1^2 \gamma$

$= 2\tau \ln \frac{n_2}{n_{Q,2} Z_{int,2}} + 2 \cdot 4\pi R_2^2 \gamma$

$\Rightarrow \ln \frac{n_1}{n_{Q,1} Z_{int,1}} = \ln \left[\left(\frac{n_2}{n_{Q,2} Z_{int,2}} \right)^2 \right] + \frac{4\pi\gamma}{\tau} (2R_2^2 - R_1^2)$

$\frac{n_1}{n_{Q,1} Z_{int,1}} = \left(\frac{n_2}{n_{Q,2} Z_{int,2}} \right)^2 \exp \left(\frac{4\pi\gamma}{\tau} (2R_2^2 - R_1^2) \right)$

(b, 1 pt.) The above reaction must conserve volume – e.g. the volume of oil in the large droplet must equal the volume of oil in the two smaller droplets. Using this, remove R_2 from your expression from part (a).

(b) Volume conservation: $\frac{4}{3} \pi R_1^3 = 2 \cdot \frac{4}{3} \pi R_2^3$
 $\Rightarrow R_1^3 = 2 R_2^3 \Rightarrow \frac{R_1}{R_2} = 2^{1/3}$

$\Rightarrow \frac{n_1}{n_{Q_1} Z_{int,1}} = \left(\frac{n_2}{n_{Q_2} Z_{int,2}} \right)^2 \exp \left[\frac{4\pi\gamma R_1^2}{\tau} (2^{1/3} - 1) \right]$

note $\frac{4\pi\gamma R_1^2}{\tau} = 2 / 2^{2/3} = 2^{1/3}$

(c, 2 pts.) Suppose n_1 is kept roughly constant, and the surface tension is nonzero and positive. If you want to **decrease** the concentration of small droplets, should you increase or decrease the surface tension? Does this result make sense? Explain.

Answer: From above, if n_1 is constant and n_2 is to decrease, $\exp(\gamma \dots)$ must increase; therefore we should **increase** the surface tension. This makes sense: If the surface tension is higher, the energetic cost of creating two smaller droplets (whose combined surface area is larger than that of one large drop, as you can verify) is greater, shifting the equilibrium away from small droplets.

(d, 2 pts.) Suppose the surface tension is **zero**. Assume that Z_1 and Z_2 are roughly similar in magnitude. (This might not be a good assumption in “real life.”) Given that these are ideal solutions, which of the following statements, if any, is true: $n_2 \gg n_1$ or $n_1 \gg n_2$. Does this result make sense? Explain.

If $\gamma = 0$, $\frac{n_1}{n_{Q_1} Z_{int,1}} = \left(\frac{n_2}{n_{Q_2} Z_{int,2}} \right)^2$
 $n_1 \ll n_{Q_1}$, and $Z_{int,1} > 1$ so $\frac{n_1}{n_{Q_1} Z_{int,1}} \ll 1$.
 Similarly $\frac{n_2}{n_{Q_2} Z_{int,2}} \ll 1$. Both are small positive numbers.
 — The square of a small positive number is an even smaller positive number (try e.g. 0.001), so: $\frac{n_2}{n_{Q_2} Z_{int,2}} \gg \frac{n_1}{n_{Q_1} Z_{int,1}}$

If we imagine $Z_{int,1}$ & $Z_{int,2}$ are roughly similar in magnitude,
 $n_2 \gg n_1$.
 This is as expected —
 in the absence of surface tension, creating lots of small droplets maximizes the entropy & lowers the free energy.