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NOTES ON MEAN FIELD THEORY, APPLIED TO THE ISING MODEL

<u>Mean Field Theory</u> (applied to the Ising Model)

Consider just <u>one</u> spin, i. Its energy $E_i = -J s_i \sum_j s_j$,

where the sum is over all neighbors, j.

* The Mean Field Idea: Replace $\sum_j s_j$ with $N_n \langle s \rangle$, where

$\langle s \rangle$ is the <u>average</u> value of s for the <u>whole system</u>.

N_n is the number of nearest neighbors.

Q: would you expect this approximation to be better in lower or higher dimensions?

Now $E_i = -J N_n s_i \langle s \rangle$. How many "states" in this system?

Just two: $s_i = +1$, with energy $-J N_n \langle s \rangle$

$s_i = -1$, " " $+J N_n \langle s \rangle$

How can we calculate $\langle s_i \rangle$?

<u>As usual</u>. $\langle s_i \rangle = \sum_{\text{states}} s_i \, p(s_i)$.

$$Z = \exp\left(\frac{J N_n \langle s \rangle}{\tau}\right) + \exp\left(-\frac{J N_n \langle s \rangle}{\tau}\right) = 2\cosh\left(\frac{J N_n \langle s \rangle}{\tau}\right)$$

$$\langle s_i \rangle = \frac{1}{Z}\left((+1)\exp\left(\frac{J N_n \langle s \rangle}{\tau}\right) + (-1)\exp\left(-\frac{J N_n \langle s \rangle}{\tau}\right)\right) = \frac{2\sinh\left(\frac{J N_n \langle s \rangle}{\tau}\right)}{Z}$$

$$\Rightarrow \quad \langle s_i \rangle = \tanh\left(\frac{J N_n \langle s \rangle}{\tau}\right)$$

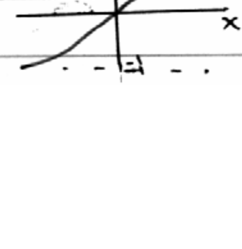
There's nothing "special" about spin i, so $\langle s_i \rangle = \langle s \rangle$!

$$\Rightarrow \quad \boxed{\langle s \rangle = \tanh\left(\frac{J N_n \langle s \rangle}{\tau}\right)}$$

A nice equation: $\langle s \rangle$ in terms of J, N_n, τ.

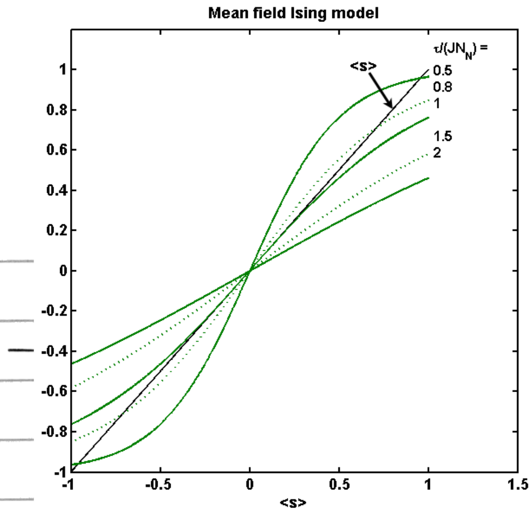
Unfortunately, transcendental. Except for trivial $\langle s \rangle = 0$, can't just solve for $\langle s \rangle$. What to do?

Numerical or <u>graphical</u> solutions. Note (recall)



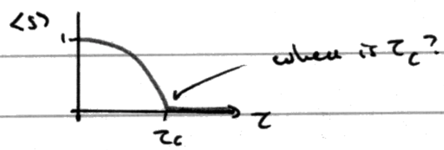
Plot the right (green) and left side (black) of the above relation, versus <s>, for various temperatures. See where the curves intersect. (Do this yourself.)

Graphs:



High τ: $\langle s \rangle = 0$ is the only solution.

Low τ: A $\langle s \rangle \neq 0$ solution! The "spontaneous" value of $\langle s \rangle$, where the curves intersect, looks like:



At $\tau = \tau_c$, green curve just skims the black @ $\langle s \rangle = 0$.

$$\Rightarrow \text{slopes equal} \Rightarrow \quad \frac{d}{d\langle s \rangle} \tanh\left(\frac{JN_n \langle s \rangle}{\tau_c}\right)\Bigg|_{\langle s \rangle = 0} = \frac{d}{d\langle s \rangle} \langle s \rangle \Bigg|_{\langle s \rangle = 0}$$

$$\frac{JN_n}{\tau_c} \operatorname{sech}^2\left(\frac{JN_n \langle s \rangle}{\tau_c}\right)\Bigg|_{\langle s \rangle = 0} = 1$$

$$JN_n / \tau_c = 1 \Rightarrow \boxed{\tau_c = JN_n}$$

* Mean-field Ising model predicts a phase transition when $k_B T$ equals the coupling ($\times$ # neighbors).

E.g. 2D square $\tau_c = 4J$.

Onsager: Exact 2D Ising $\tau_c = 2.27\,J$.

A sharp phase transition! ★