Due date: Wednesday, April 9, 5pm. (Turn in to the assignment to the box outside my door.)
Reading: Kittel & Kroemer Chapter 8 through p. 244 (Irreversible work).

1 (4 pts.) Refrigerators. In class, we considered the operation of a heat engine – a device that uses heat to perform work. A refrigerator is essentially a heat engine operated in reverse. We have a cold reservoir at temperature $\tau_C$ and a hot reservoir at $\tau_H$. The cold reservoir is the inside of your kitchen fridge, for example, and the hot reservoir is the room. We perform work $W$ in order to extract heat $Q_C$ from the cold reservoir\(^1\). Some waste heat $Q_H$ is dumped into the kitchen. See the diagram. We define the “coefficient of performance” ($\gamma$) of the refrigerator as $\gamma \equiv \frac{Q_C}{W}$. The higher the COP, the less work it takes to extract a given amount of heat.

(a, 3 pts.) Determine the fundamental limits on the performance of a refrigerator – show that $\gamma \leq \frac{\tau_C}{\tau_H - \tau_C}$. Proceed as we did for the heat engine, applying the thermodynamic identity to a cyclic process and noting that the entropy “leaving” the machinery must be at least equal to the entropy “entering” it. Note, by the way, that $\gamma$ is not bounded by 1, and is largest if the reservoir temperatures are similar.

(b, 1 pt.) For a kitchen refrigerator, estimate the maximum possible $\gamma$. (Use reasonable values for the reservoir temperatures, and don’t forget that we always need to deal with the absolute temperature.) In practice, of course, $\gamma$ is considerably less than this.

2 (7 pts.) An ideal diatomic gas. This problem may look familiar... For a diatomic molecule, (e.g. “dumbbells” like nitrogen, $N_2$), there are two internal degrees of freedom, corresponding to vibration along the dumbbell axis and rotation perpendicular to the axis. This means there are five total degrees of freedom, since the diatomic molecule has the same ability to move (translate) in three dimensions as does a monatomic molecule.

(a, 1 pt.) Using the Equipartition Theorem: What is the energy, $U$, of the gas of $N$ diatomic molecules as a function of the number of particles, $N$, and the temperature, $\tau$?

(b, 1 pt.) Show that in general for an ideal gas with $d$ degrees of freedom, $dU = c_V d\tau$, and determine $c_V$ in terms of $d$ and $N$.

(c, 5 pts.) Defining $\gamma$ as $\frac{c_p}{c_V}$, the ratio of the specific heat capacities at constant pressure and constant volume, show that $\gamma$ for a diatomic ideal gas is $7/5$. The work done by a change in volume $dV$ is $PdV$, just as for a monatomic gas (as it must be, since this is just the mechanical work associated with a weight on a piston, for example).

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\(^1\) Our analysis applies to any refrigeration scheme. We won’t explore the methods by which actual refrigerators work. Basically, your kitchen refrigerator pumps and compresses liquids and gases to transfer heat.
3 (11 pts.) The Otto cycle. In class we introduced the Otto cycle, used in internal combustion engines. Here we will calculate the efficiency of an ideal Otto cycle, acting on an ideal gas. The $P-V$ diagram is shown; the engine cycles between points $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$...

- Gasoline vapor and air are compressed, isentropically ($1 \rightarrow 2$).
- A spark causes rapid combustion at roughly constant volume ($2 \rightarrow 3$). Heat $Q_H$ is absorbed from the burning fuel.
- The gas expands ("power stroke") isentropically ($3 \rightarrow 4$).
- Heat is expelled to reservoirs / exhaust ($4 \rightarrow 1$), at constant $V$.

(We’re glossing over many details of engine design, but they don’t concern us.) Consider the working substance to be an ideal gas, so $PV = N\tau$. The gas may have internal degrees of freedom (e.g. rotational modes) characterized by some particular $\gamma$, where $\gamma \equiv e_p / e_v$.

(a, 2 pts.) Call the temperatures at points 1, 2, 3, and 4 $\tau_1, \tau_2, \tau_3,$ and $\tau_4$, respectively. What temperature is hottest? What is coldest? You may find it helpful to consider each “leg” of the cycle separately and consider which of its endpoints is hotter or colder.

(b, 2 pts.) What is the heat input, $Q_H$, for the engine? Express your answer in terms of $c_v$ and two of the temperatures ($\tau_1, \tau_2, \tau_3$, and $\tau_4$). *Hints: Only one leg of the cycle is relevant; use your answer from (2b).*

(c, 1 pt.) What is the heat output, $Q_L$, for the engine? Express your answer in terms of $c_v$ and two of the temperatures ($\tau_1, \tau_2, \tau_3,$ and $\tau_4$).

(d, 2 pts.) Calculate the efficiency, $\eta$, of the engine using your answers to (c) and (d). Leave your expression in terms of $\tau_1, \tau_2, \tau_3,$ and $\tau_4$. *Hint: You can do this without calculating the work done. In fact, it’s easier if you don’t calculate $W$. How can you remove $W$ from the definition of $\eta$? (Of course, there’s nothing wrong with calculating $W$).*

(e, 2 pts.) Consider a Carnot cycle operating with reservoirs at the hottest and coldest temperatures of this Otto Cycle (see part (a)). Which is more efficient, the Carnot or Otto cycle? Explain.

(f, 3 pts.) We learned last term how temperature and volume are related for adiabatic expansions.

Using this, show that the efficiency of the Otto cycle is $\eta = 1 - r^{\gamma - 1}$, where the “compression ratio” $r = V_2 / V_1$, the ratio of the volumes of the constant-volume stages.

(g, 1 pt) Look up a typical value of $r$ for a car’s engine and determine the ideal Otto cycle efficiency this gives. Note Problem 2; air is mostly diatomic gas. In practice, by the way, $\eta$ for a car is about 20-25%. (Cite your source for $r$; a quick search on the internet is fine.).

4 (8 pts.) A heat engine and a refrigerator. The efficiency of a heat engine is greater if the temperature difference between the reservoirs is greater, Mr. K. learns. “Eureka!” he says. “Instead of running my heat engine between reservoirs at temperature $\tau_H$ and room temperature, $\tau_R$, I’ll use a refrigerator to cool the cold reservoir to temperature $\tau_F$. Brilliantly, I’ll just use some of the work output by the heat engine to power the fridge! I’ll improve the efficiency of my heat engine system!” What is wrong with this scheme? Calculate its efficiency and compare to that of the heat engine alone. (Consider ideal, Carnot efficiencies.)

This problem is more involved than its short length might imply. Some suggestions:
• Make a clear diagram of the heat and work flows. The heat engine and the refrigerator share the “cold reservoir” at \( \tau_p \).

• You may wish to call the output work \( W_o \) and the work used by the fridge \( W_x \). Apply conservation of energy to the heat engine and the fridge, separately. One way to proceed is to combine these expressions to eliminate \( W_x \).

• Note that the efficiency we’re trying to calculate is \( \eta = \frac{W_o}{Q_h} \), where \( Q_h \) is the heat flow from the hot reservoir to the heat engine. Derive an expression for \( \eta \) in terms of temperatures and heats.

• Compare this efficiency to the Carnot efficiency for the system without the fridge — i.e. operating between \( \tau_h \) and \( \tau_l \). Is it larger or smaller? Note the following: you’ll have to think of the “cold reservoir” as a finite system, rather than an “ideal” reservoir that can input or output arbitrary amounts of heat. Another way of saying this is that during a cyclic process, the entropy “created” by heat flow from the cold reservoir must be at least equal to the entropy “destroyed” by heat flow into the cold reservoir.