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Physics 353: Problem Set 1 – SOLUTIONS

1 Refrigerators

(a) $\not{d}Q = dU + \not{d}W$. As usual, consider a cyclic process; $\oint dU = 0$, so $\oint \not{d}Q = \oint \not{d}W$, so $Q_H - Q_C = W$ (see diagram in the assignment). Cyclic, so the entropy of the engine is unchanged (at best) or increases due to friction, over one cycle.

Entropy "in" = Q_C / τ_C, since $Q = \tau\, \Delta\sigma$. Entropy "out" = Q_H / τ_H .

At best, $Q_H / \tau_H = Q_C / \tau_C$. Or more generally, $Q_H / \tau_H \geq Q_C / \tau_C$, so $Q_H / Q_C \geq \tau_H / \tau_C$

The COP $\gamma \equiv {Q_C}/{W}$, so $\gamma = \dfrac{Q_C}{Q_H - Q_C} = \dfrac{1}{\frac{Q_H}{Q_C} - 1}$, so $\gamma \leq \dfrac{1}{\frac{\tau_H}{\tau_C} - 1}$, i.e. $\boxed{\gamma \leq \dfrac{\tau_C}{\tau_H - \tau_C}}$.

(b) For a kitchen refrigerator, $\tau_C \approx 4^o C = 277K$, and $\tau_H \approx 23^o C = 296K$. Therefore the max. COP we'd expect is about 15.

2 An ideal diatomic gas.

(a) 5 degrees of freedom, and, via the Equipartition Theorem, energy $\frac{1}{2}\tau$ per particle per degree of freedom, so $U = \frac{5}{2} N\tau$.

(b) In general, $U = \frac{d}{2} N\tau$. By definition, $c_V \equiv \left.\dfrac{dQ}{d\tau}\right|_V$. At constant volume, $dW = PdV = 0$, so from the thermodynamic identity $\not{d}Q = dU$. Therefore $c_V \equiv \left.\dfrac{dQ}{d\tau}\right|_V = \dfrac{dU}{d\tau} = \dfrac{d}{2} N$.

(c)

$c_V \equiv \left.\dfrac{dQ}{d\tau}\right|_V$. $\not{d}Q = dU + \not{d}W$.

$= dU + P\,dV$

$\rightarrow c_V = \left.\dfrac{dQ}{d\tau}\right|_V = \left.\dfrac{dU}{d\tau}\right|_V = \dfrac{d}{d\tau}\left(\frac{5}{2} N\tau\right) = \frac{5}{2} N$.

$c_P \equiv \left.\dfrac{dQ}{d\tau}\right|_P = \left.\dfrac{d}{d\tau}(dU + P\,dV)\right|_P = \frac{5}{2} N + P \left.\dfrac{dV}{d\tau}\right|_P$.

$PV = N\tau$, so $\dfrac{dV}{d\tau} = \dfrac{N}{P}$. $\rightarrow c_P = \frac{5}{2} N + N = \frac{7}{2} N$.

$\Rightarrow \gamma \equiv c_P / c_V = \dfrac{7/2\, N}{5/2\, N} = \underline{\gamma = \dfrac{7}{5}}$.

3 The Otto cycle.

(a)

$PV = N\tau$. (Ideal gas).

Consider point 1 → point 2. isentropic compression, so τ rises. $\tau_2 > \tau_1$

$2 \to 3$. V = const, P rises, so τ rises by the Ideal Gas law: $\tau_3 > \tau_2$.

$3 \to 4$. opposite of $1\text{-}2$ → τ drops: $\tau_4 < \tau_3$

$4 - 1$. V const, P drops → $\tau_1 < \tau_4$

$\Rightarrow$ $\tau_1 < \tau_2 < \tau_3$

$\tau_1 < \tau_4 < \tau_3$.

$\boxed{\text{Minimum } \tau \text{ is } \tau_1 \quad \text{Maximum } \tau \text{ is } \tau_3}$

(b) Heat is absorbed during $2 \to 3$, by construction. (Also, $1\text{-}2$ and $3\text{-}4$ have no heat flow anyway, since $\Delta\sigma = 0$). What is $Q_{2\to3}$?

$đQ = dU + đW$; $đQ = dU + P\,dV$.

$Q_{2\to3} = \int_2^3 dU + \int_2^3 P\,dV$. V is constant on this leg, so this term is zero!

$Q_{2\to3} = \int_2^3 dU = \int_{\tau_2}^{\tau_3} C_V\, d\tau$, since U only depends on τ.

→ $Q_{2\to3} = \boxed{Q_H = \boxed{C_V(\tau_3 - \tau_2)}}$

(c) Similarly, $\boxed{Q_L = C_V(\tau_4 - \tau_1)}$ (By definition, $Q_L > 0$ for "outward" heat flow.

(d) $\eta \equiv W / Q_H$. From the thermodynamic identity, $W = Q_H - Q_L$, so $\eta = \dfrac{Q_H - Q_L}{Q_H} = 1 - \dfrac{Q_L}{Q_H}$. Via parts b and c, $\boxed{\eta = 1 - \dfrac{\tau_4 - \tau_1}{\tau_3 - \tau_2}}$.

(e) [*deleted*] This is a somewhat subtle point, and you may skip this. Here's a non-mathematical discussion. We imagined that the only heat input, and hence the only "entropy gain" was that of the gas (working substance) during stage 2-3. However, this can't be true, even ideally. *Unlike* the Carnot or Stirling cycles, temperature is not constant during this stage. Therefore, we must go from one temperature (thermal equilibrium) to another. We know that whenever we put two systems together to equilibrate them, the total entropy of the supersystem increases. This was not accounted for in our $\Delta\sigma$, and so we under-counted Q_H, and so our η is larger than what would be possible for the Carnot cycle.

(f) For adiabatic expansions, $\tau V^{\gamma-1} = \text{const.}$

Therefore $\tau_1 V_2^{\gamma-1} = \tau_2 V_1^{\gamma-1}$

$$\tau_3 V_1^{\gamma-1} = \tau_4 V_2^{\gamma-1}$$

$$\Rightarrow \quad \eta = 1 - \frac{\tau_3\left(\frac{V_1}{V_2}\right)^{\gamma-1} - \tau_2\left(\frac{V_1}{V_2}\right)^{\gamma-1}}{\tau_3 - \tau_2}$$

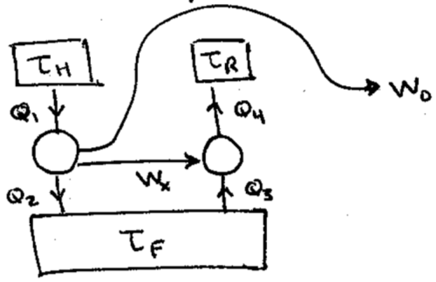
$$\Rightarrow \eta = 1 - \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

$r \equiv \dfrac{V_2}{V_1}$, so $\underline{\eta = 1 - r^{1-\gamma}}$

(g) A typical automobile compression ratio is $r \approx 10$ (e.g. *http://en.wikipedia.org/wiki/Compression_ratio*). With $\gamma = 7/5$, this means $\eta \approx 0.6$.

4 A heat engine and a refrigerator.

Making a clean diagram is very helpful:



W_o = "output" work

heat engine, running between τ_H & τ_F

Fridge, running between τ_R & τ_F. Input work $= W_x$.

Conservation of energy (cyclic processes): $Q_1 = W_o + W_x + Q_2$ (heat engine)

$$W_x + Q_3 = Q_4 \qquad \text{(fridge)}$$

Entropy in = Entropy out : $\dfrac{Q_1}{\tau_H} = \dfrac{Q_2}{\tau_F}$; $\dfrac{Q_4}{\tau_R} = \dfrac{Q_3}{\tau_F}$

we want to figure out the efficiency $\eta =$ work out/heat input

$$\eta = W_o/Q_1 .$$

$$W_o = Q_1 - Q_2 - W_x = Q_1\left(1 - \frac{\tau_F}{\tau_H}\right) - W_x \qquad \text{(heat engine)}$$

$$W_x = Q_4 - Q_3 = Q_3\left(\frac{\tau_R}{\tau_F} - 1\right) \qquad \text{(fridge)}$$

Combine: $W_o = Q_1(1 - \tau_F/\tau_H) - Q_3\left(\dfrac{\tau_R}{\tau_F} - 1\right)$.

$$\eta = \frac{W_o}{Q_1} = \left(1 - \frac{\tau_F}{\tau_H}\right) - \frac{Q_3}{Q_1}\left(\frac{\tau_R}{\tau_F} - 1\right)$$

. without the fridge,

$$\eta_C = 1 - \frac{\tau_R}{\tau_H} \quad \text{(Carnot)}$$

Is this efficiency better or worse than the Carnot efficiency?

i.e. is $\eta - \eta_c > 0$ or < 0?

$$\eta - \eta_c = \left(1 - \frac{\tau_F}{\tau_H}\right) - \frac{Q_3}{Q_1}\left(\frac{\tau_R}{\tau_F} - 1\right) - \left(1 - \frac{\tau_R}{\tau_H}\right)$$

$$= \left(\frac{\tau_R - \tau_F}{\tau_H}\right) + \frac{Q_3}{Q_1}\left(1 - \frac{\tau_R}{\tau_F}\right) = \left(\frac{\tau_R - \tau_F}{\tau_H}\right) - \frac{Q_3}{Q_1}\left(\frac{\tau_R - \tau_F}{\tau_F}\right)$$

$$= \eta - \eta_c = \underbrace{\left(\frac{\tau_R - \tau_F}{\tau_H}\right)}_{>0,\ \text{since } \tau_R > \tau_F} \underbrace{\left(1 - \frac{Q_3}{Q_1}\frac{\tau_H}{\tau_F}\right)}_{\text{is this } >0 \text{ or } <0?}$$

$$\frac{Q_1}{\tau_H} = \frac{Q_2}{\tau_F}, \text{ from above}, \quad \text{so} \quad \eta - \eta_c = \left(\frac{\tau_R - \tau_F}{\tau_H}\right)\left(1 - \frac{Q_3}{Q_2}\cancelto{1}{\frac{\tau_F}{\tau_F}}\right)$$

$$\rightarrow \eta - \eta_c = \left(\frac{\tau_R - \tau_F}{\tau_H}\right)\left(1 - \frac{Q_3}{Q_2}\right).$$

For our system to be reversible, the entropy "out of" the cold reservoir must equal the entropy "in." More generally, entropy does not decrease during the cyclic process – at best it stays the same. Equivalently, the heat we extract from the cold reservoir must be at least as large as the heat put in if the temperature of the cold reservoir is to be constant. All of these are ways of saying: $\frac{Q_3}{\tau_F} \geq \frac{Q_2}{\tau_F}$, and therefore $Q_3 \geq Q_2$. Therefore $1 - \frac{Q_2}{Q_3} \geq 0$, and $\eta - \eta_C \leq 0$, i.e. $\boxed{\eta \leq \eta_C}$.