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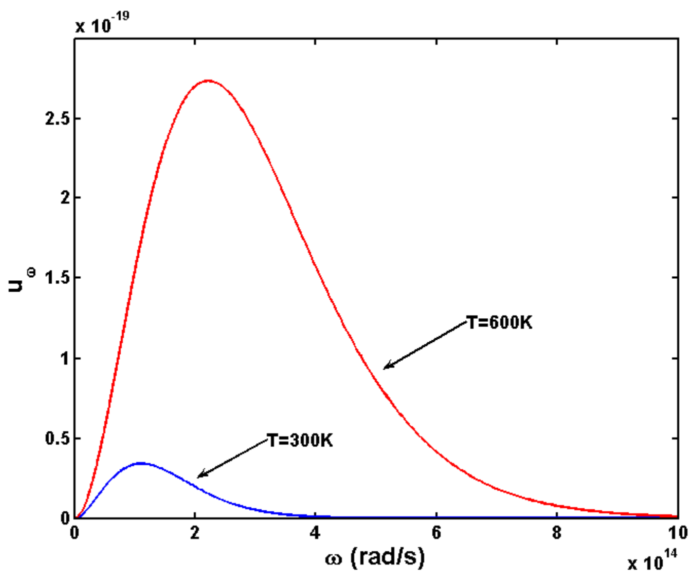
University of Oregon; Spring 2008

Physics 353: Problem Set 2 – SOLUTIONS

1 Graphs



2 The blackbody peak.

2a. $u_\omega = \frac{dU}{V\,d\omega} = \frac{3\hbar}{2\pi^2 c^3} \frac{\omega^3}{(e^{\hbar\omega/\tau} - 1)}$.

Peak? define $x \equiv \frac{\hbar\omega}{\tau}$ (dimensionless).

$\rightarrow u_\omega = \underbrace{\frac{3\hbar}{2\pi^2 c^3}\left(\frac{\tau}{\hbar}\right)^3}_{} \frac{x^3}{e^x - 1}$.

$\frac{du_\omega}{dx} = (\ )\left[\frac{3x^2}{e^x - 1} + \frac{-x^3 e^x}{(e^x-1)^2}\right]$. $= 0$ if

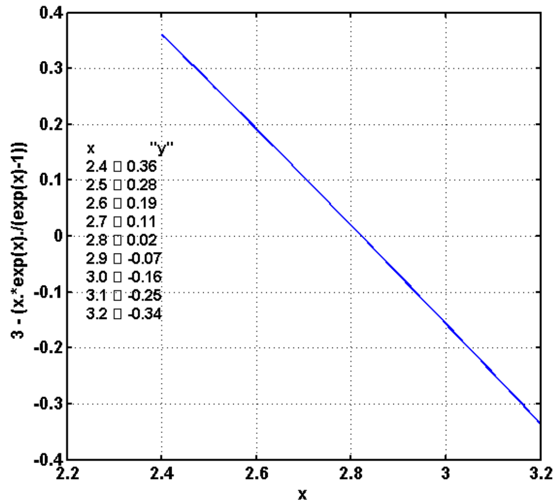
$\frac{3x^2}{e^x-1} - \frac{x^3}{(e^x-1)^2} = 0 \rightarrow \underline{3 - \frac{x e^x}{e^x - 1}} = 0$

(ignoring the trivial $x=0$ solution)

where is this function zero? Graph or plot. (call this function "y")

We see (below) that it's zero @ $x = 2.8$[redacted],

or $\frac{\hbar\omega}{\tau} = 2.8$[redacted], i.e. $\underline{\omega = 2.8 \frac{\tau}{\hbar}}$.



(b)

We want the spectral density in terms of $\underline{\lambda}$, not ω.

Note $\lambda f = c$, so $\lambda = \frac{c}{f} = \frac{2\pi c}{\omega}$; $\omega = \frac{2\pi c}{\lambda}$

<u>Not quite correct</u> : $u_\lambda = \frac{3\hbar}{2\pi^2 c^3} \frac{(2\pi c)^3}{\left[\lambda^3 \left(\exp\left(\frac{\hbar}{\tau}\frac{2\pi c}{\lambda}\right) - 1\right]}$

Look for the maximum of this function.

Can proceed as above, or note that

$\frac{du}{d\lambda} = \frac{du}{d\omega}\frac{d\omega}{d\lambda}$; $\frac{d\omega}{d\lambda} = \frac{-2\pi c}{\lambda^2} \neq 0$, so

$\frac{du}{d\lambda} = 0$ at the same ω for which $\frac{du}{d\omega} = 0$,

i.e. $\omega = \frac{\tau}{\hbar} 2.8$, so peak at $\underline{\lambda = \frac{2\pi c \hbar}{2.8\,\tau}}$.

This is <u>wrong</u>. Why?

Correct: what $\underline{is}$ u_λ? Spectral density in terms of $\underline{\lambda}$.

i.e. how much energy in some range $d\lambda$.

$$u_\lambda = \frac{dU}{V\, d\lambda}. \qquad dU = V \frac{3\hbar}{2\pi^2 c^3} \frac{\omega^3\, d\omega}{e^{\hbar\omega/\tau} - 1}.$$

$\omega = \frac{2\pi c}{\lambda}$ (since $\lambda f = c$), so $d\omega = \frac{-2\pi c}{\lambda^2} d\lambda$.

$$dU = V \frac{3\hbar}{2\pi^2 c^3} \frac{(2\pi c)^3}{\lambda^3} \frac{(-) 2\pi c\, d\lambda}{\lambda^2 \left(\exp\left(\frac{\hbar}{\tau}\frac{2\pi c}{\lambda}\right) - 1\right)}$$

(minus sign just means λ is "backwards" w.r.t. ω.)

$$\rightarrow u_\lambda = \frac{dU}{V d\lambda} = \frac{3\hbar}{2\pi^2 c^3} (2\pi c)^4 \frac{1}{\lambda^5 \left[\exp\left(\frac{\hbar 2\pi c}{\tau\lambda}\right) - 1\right]}.$$

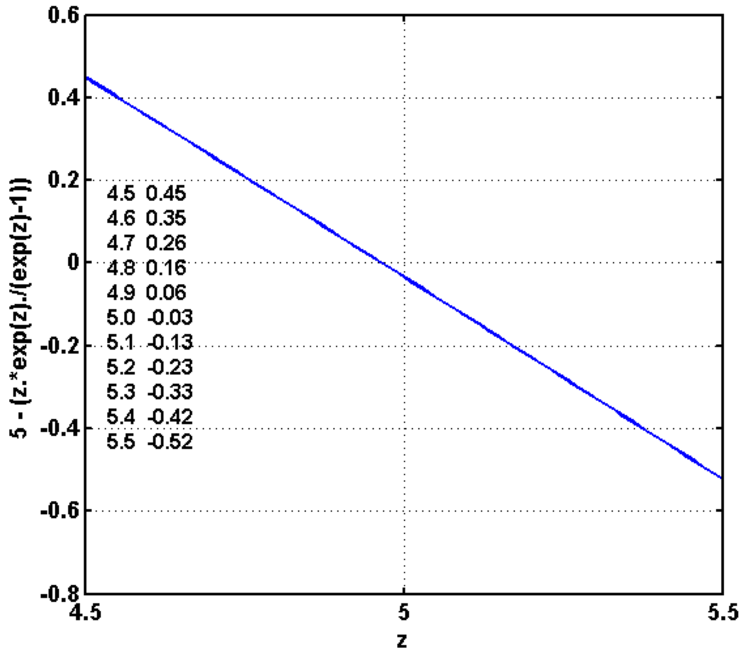
Dimensionless $z \equiv \frac{2\pi c \hbar}{\tau\lambda}$. We're looking for the max. of $\frac{z^5}{e^z - 1}$. $\frac{d}{dz}\left(\frac{z^5}{e^z-1}\right) = \frac{5z^4}{e^z - 1} + \frac{-z^5 e^z}{(e^z-1)^2} = 0$

$$\rightarrow 5 - \frac{z e^z}{e^z - 1} = 0 \rightarrow$$

Again solve graphically or numerically ... **plot below**

$$\rightarrow \quad \underline{z = 5.0} \rightarrow \frac{2\pi c\hbar}{\tau\lambda} = 5.0 \rightarrow \boxed{\lambda = \frac{2\pi c\hbar}{5.0\,\tau}}$$

We should $\underline{\text{expect}}$ that the peak in u_λ is at a different "location" than the peak in ω since the transformation $\lambda \sim 1/\omega$ "stretches" different parts of the function differently.



3 Numbers of photons

(a) We showed in class that the number of photons with ω between ω and $\omega + d\omega$ is

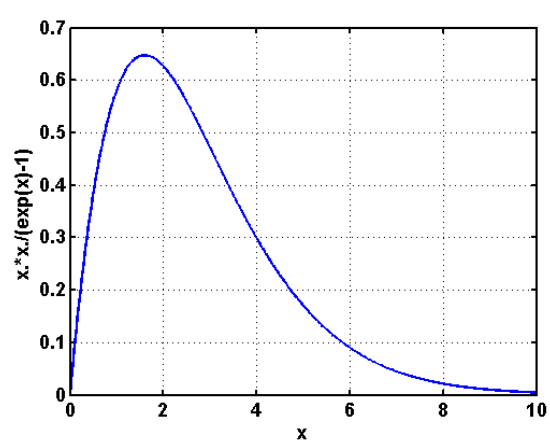
$$dN = \langle n \rangle d\Omega_0 = \frac{V\omega^2}{\pi^2 c^3}\left(\frac{d\omega}{\exp\left(\frac{\hbar\omega}{\tau}\right)-1}\right).$$

N is simply the integral of this. Defining $x \equiv \hbar\omega/\tau$ as usual,

$$N = \int_0^\infty \frac{V\omega^2}{\pi^2 c^3}\left(\frac{d\omega}{\exp\left(\frac{\hbar\omega}{\tau}\right)-1}\right) = \int_0^\infty \frac{V\tau^3 x^2}{\pi^2 \hbar^3 c^3}\left(\frac{dx}{\exp(x)-1}\right),$$

so

$$N = \frac{V}{\pi^2}\left(\frac{\tau}{\hbar c}\right)^3 \int_0^\infty \left(\frac{x^2 dx}{\exp(x)-1}\right)$$



Graphically (a good, quick way to do evaluate the integral) we see that the area under the curve covers about 13 boxes of area 0.2, so the integral ≈2.6.

Numerically integrating, the integral ≈2.4.

I'm sure you could look this up somewhere, also.

(b) Plugging in numbers, $\frac{V}{\pi^2}\left(\frac{\tau}{\hbar c}\right)^3 = 8.9\times10^{13}$, so $N = 2.1\times10^{14}$ photons.

(c) Ideal gas: $N = \frac{PV}{\tau}$. Equating this with $N = \frac{V}{\pi^2}\left(\frac{\tau}{\hbar c}\right)^3 \int_0^\infty \left(\frac{x^2 dx}{\exp(x)-1}\right)$, we need $\frac{P}{\tau} = \frac{V}{\pi^2}\left(\frac{\tau}{\hbar c}\right)^3 2.4$, where I've inserted the value of the integral. Therefore $\frac{P(\hbar c)^3 \pi^2}{2.4} = \tau^4$. Noting that $\tau = k_B T$, this comes out to be about 140,000 K.

4 There are many ways to go about expressing the partition function for the photons. Here is one approach that I like. Think of one particular configuration of the system that has n_1 photons in mode 1, n_2 photons in mode 2, n_3 photons in mode 1, etc., where mode 1 has frequency ω_1, mode 2 has frequency ω_2, etc. What is the energy of this state?

$$E(n_1, n_2, n_3, \ldots) = n_1\hbar\omega_1 + n_2\hbar\omega_2 + n_3\hbar\omega_3 + \cdots$$

$$\rightarrow Z = \sum_{n_1}\sum_{n_2}\sum_{n_3}\cdots \exp\left(-\frac{n_1\hbar\omega_1 + n_2\hbar\omega_2 + n_3\hbar\omega_3 + \cdots}{\tau}\right)$$

$$= \sum_{n_1}\sum_{n_2}\sum_{n_3}\cdots \exp\left(-\frac{n_1\hbar\omega_1}{\tau}\right)\exp\left(-\frac{n_2\hbar\omega_2}{\tau}\right)\exp\left(-\frac{n_3\hbar\omega_3}{\tau}\right)\cdots$$

$$= \sum_{n_1}\exp\left(\frac{-n_1\hbar\omega_1}{\tau}\right)\left[\sum_{n_2}\sum_{n_3}\cdots \exp\left(\frac{-n_2\hbar\omega_2}{\tau}\right)\exp\left(\frac{-n_3\hbar\omega_3}{\tau}\right)\cdots\right],$$

since all the "n_1 factors" are constants as far as the other sums are concerned

$$= \left[\sum_{n_1}\exp\left(\frac{-n_1\hbar\omega_1}{\tau}\right)\right]\left[\sum_{n_2}\exp\left(\frac{-n_2\hbar\omega_2}{\tau}\right)\right]\left[\sum_{n_3}\cdots\right]$$

similarly. Note that all these sums are the same!

$$\rightarrow Z = \prod_m \left[\underbrace{\sum_n \exp\left(\frac{-n\hbar\omega_m}{\tau}\right)}_{\text{geom. series, } \frac{1}{1-e^{-\hbar\omega_m/\tau}}}\right]$$

$$\rightarrow Z = \prod_m \frac{1}{1-\exp\left(\frac{-\hbar\omega_m}{\tau}\right)}$$

(b) $F = -\tau \ln Z$

$$= -\tau \ln\left(\prod_m \frac{1}{1-e^{-\hbar\omega_m/\tau}}\right)$$

$$= \tau \sum_m \ln\left(1-e^{-\hbar\omega_m/\tau}\right)$$

(c) Note that we can easily turn the above sum over modes into an integral over modes – i.e. over "$d\Omega_0$". But we'd like to make this an integral over frequency, since all our expressions are functions of frequency. We already figured out how many modes are in a given frequency range – i.e. what $d\Omega_0 / d\omega$ is (see your lecture notes): $d\Omega_0 / d\omega = \frac{V}{\pi^2 c^3}\omega^2 d\omega$. Therefore:

$$F = \tau \int_0^\infty \frac{V}{\pi^2 c^3} \omega^2 d\omega \cdot \ln\left(1-e^{-\hbar\omega/\tau}\right)$$

$$= \frac{V\tau}{\pi^2 c^3} \int_0^\infty \ln\left(1-e^{-\hbar\omega/\tau}\right) \omega^2 d\omega$$

$$= \frac{V\tau}{\pi^2 c^3} \cdot \frac{\tau^3}{\hbar^3} \int_0^\infty \ln\left(1-e^{-x}\right) x^2 dx$$

$$= \frac{V\tau^4}{\pi^2 c^3 \hbar^3} \cdot \left(\frac{x^3}{3} \ln(1-e^{-x}) \Big|_0^\infty - \frac{1}{3} \int_0^\infty \frac{x^3 dx}{e^x - 1} \right)$$

$$= -\frac{V\tau^4}{\pi^2 c^3 \hbar^3} \cdot \frac{1}{3} \cdot 6 \times \frac{\pi^4}{90}$$

$$= -\frac{\pi^2 V \tau^4}{45 \hbar^3 c^3}$$

(d) (Yan writes using $T = \tau / k_B$ and $S = k_B\sigma$.)

(d). $S = k\left(\ln Z - \beta \frac{\partial}{\partial\beta} \ln Z\right)$

$$= \frac{1}{T}(-F) - \frac{1}{T}\frac{\partial}{\partial\beta} \ln Z$$

$$= \frac{\pi^2 V k^4 T^3}{45 \hbar^3 c^3} - \frac{1}{T}\frac{\partial}{\partial\beta}\left(\frac{\pi^2 V}{45 \hbar^3 c^3 \beta^3}\right)$$

$$= \frac{\pi^2 V k^4 T^3}{45 \hbar^3 c^3} + 3 \frac{1}{T} \cdot \frac{\pi^2 V k^4 T^4}{45 \hbar^3 c^3}$$

$$= \frac{4\pi^2 V k^4}{45 \hbar^3 c^3} T^3$$

(e). $P = +\tau \frac{\partial}{\partial V} \ln Z = \frac{\partial}{\partial V}(-F) = \frac{\pi^2 \tau^4}{45 \hbar^3 c^3}$.

(f) $U = -\frac{\partial}{\partial\beta} \ln Z$

$$= -\frac{\partial}{\partial\beta} \cdot \left(\frac{\pi^2 V}{45 \hbar^3 c^3 \beta^3}\right)$$

$$= \frac{3\pi^2 V \tau^4}{45 \hbar^3 c^3}$$

$$\therefore P = \frac{U}{3V}$$

5 A one-dimensional blackbody.

. A one-dimensional blackbody

the number of states with frequencies $\omega \to \omega + d\omega$ is

$$\frac{2L}{h} dp = \frac{2L}{c \cdot 2\pi} d\omega = \frac{L}{c\pi} d\omega.$$

so the number of photons is

$$\frac{L}{c\pi} \frac{d\omega}{e^{\hbar\omega/\tau} - 1}$$

$$\therefore U(\omega, T) d\omega = \frac{L}{c\pi} \cdot \frac{\hbar\omega \, d\omega}{e^{\frac{\hbar\omega}{\tau}} - 1}$$

$$\therefore U(\omega, \tau) = \frac{L\tau^2}{c\pi\hbar} \int_0^\infty \frac{\frac{\hbar\omega}{\tau} \cdot d\left(\frac{\hbar\omega}{\tau}\right)}{e^{\frac{\hbar\omega}{\tau}} - 1}$$

$$= \frac{L\tau^2}{c\pi\hbar} \int_0^\infty \frac{x\,dx}{e^x - 1}$$

$$= \frac{L\tau^2}{c\pi\hbar} \cdot \frac{\pi^2}{6}$$

$$= \frac{L\pi}{6c\hbar} \tau^2$$

$$\therefore U \propto \tau^p \quad p = 2.$$

6 (3 pts.**) Three short literature searches.**

(a) I found 229 results, nearly all of which seem to be books.

(c) I found 22,306 articles!