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Physics 353: Problem Set 3 – SOLUTIONS

1 Kittel and Kroemer Chp. 4 #18. Isentropic expansion of a photon gas.

(a) $\tau_i V_i^{1/3} = \tau_f V_f^{1/3}$. $\quad V \propto r^3$

$\tau_f = 2.7\,K \quad \tau_i = 3000\,K$

$\therefore \tau_i r_i = \tau_f r_f$

$\therefore \dfrac{r_f}{r_i} = \dfrac{\tau_i}{\tau_f} = \dfrac{3000\,K}{2.7\,K} = 1111$

(b) $dQ = 0$

$\therefore dW = -dU \qquad W = -\int dU$

$\therefore W = -(U_{final} - U_{initial})$

$= \dfrac{\pi^2}{c^3\hbar^3 15}\left(V_i \tau_{initial}^4 - V_f \tau_{final}^4\right)$

$\tau_i V_i^{1/3} = \tau_f V_f^{1/3}$
$\tau_i^3 V_i = \tau_f^3 V_f$

$= \dfrac{\pi^2}{15 c^3 \hbar^3} V_i \tau_i^3 (\tau_i - \tau_f)$

2 Phonon heat capacity.

$$U = V \frac{3\hbar}{2\pi^2 c_s^3} \int_0^{\omega_{max}} \frac{\omega^3 d\omega}{e^{\hbar\omega/\tau} - 1}$$

Low τ. "$\omega_{max} \approx \infty$". $\quad x \equiv \frac{\hbar\omega}{\tau}$.

$$U \approx V \frac{3\hbar}{2\pi^2 c_s^3} \left(\frac{\tau}{\hbar}\right)^4 \int_0^{\infty} \frac{x^3 dx}{e^x - 1}$$

$$c_s^3 = \frac{\omega_{max}^3 V}{6\pi^2 N} \text{ (class)} \qquad \text{integral} = \frac{\pi^4}{15} \text{ (class)}$$

$$\rightarrow U \approx \cancel{V} \frac{3\tau^4}{2\cancel{\pi^2}\hbar^3} \frac{6\cancel{\pi^2} N}{\omega_{max}^3 \cancel{V}} \frac{\pi^4}{15}$$

Therefore $c_V = \left.\frac{\partial U}{\partial \tau}\right|_V = \frac{12}{5}\pi^4 N \left(\frac{\tau}{\Theta_D}\right)^3$, simplifying the algebra and using the definition of the Debye temperature $\Theta_D \equiv \hbar\omega_{max}$.

2b

(b) High τ. $\quad \exp\left(\frac{\hbar\omega}{\tau}\right) \approx 1 + \frac{\hbar\omega}{\tau}$, $\quad \frac{\omega^3}{e^{\hbar\omega/\tau} - 1} \approx \frac{\omega^3}{\hbar\omega/\tau} = \frac{\omega^2\tau}{\hbar}$

So $U \approx V \frac{3\hbar}{2\pi^2 c_s^3} \int_0^{\omega_{max}} \frac{\omega^2 \tau}{\hbar} d\omega = \frac{V \cancel{3\hbar}\tau}{2\pi^2 c_s^3 \cancel{\hbar}} \frac{\omega_{max}^3}{\cancel{3}}$

From class $\omega_{max} = (6\pi^2 c_s^3 N/V)^{1/3}$,

so $U \approx \frac{\cancel{3}\cancel{V}}{2\cancel{\pi^2}\cancel{c_s^3}} \tau \, 6\cancel{\pi^2}\cancel{c_s^3} \frac{N}{\cancel{V}} = 3N\tau$.

$C_V \equiv \left.\frac{\partial Q}{\partial \tau}\right|_V = \left.\frac{dU}{d\tau}\right|_V = 3N$ ✓

[since $PdV = 0$

3 Photon Energy.

a. isothermal stage. $đQ = đU + đW = đU + P\,dV$.

$$đQ = \left(\frac{\partial U}{\partial \tau} d\tau + \frac{\partial U}{\partial V} dV\right) + P\,dV \quad \text{(general)}$$

$$đQ = \frac{\partial U}{\partial V} dV + P\,dV \quad \text{(isothermal step)}$$

b. Work done over the entire cycle $W = dP\,dV$.

Heat = heat input in part (a). $\rightarrow \eta = \frac{W}{Q_H}$

$$= \eta = \frac{dP\,dV}{\frac{\partial U}{\partial V} dV + P\,dV} = \frac{dP}{\frac{\partial U}{\partial V} + P}.$$

Carnot: $\eta = 1 - \frac{\tau_L}{\tau_H}$ (ideal) $= \frac{d\tau}{\tau + d\tau}$

$$\Rightarrow \frac{W}{Q_H} = \frac{d\tau}{\tau + d\tau}.$$

c. ... $\Rightarrow \frac{dP}{\frac{\partial U}{\partial V} + P} = \frac{d\tau}{\tau + d\tau} \Rightarrow dP\,(\tau + d\tau) = d\tau\left(\frac{\partial U}{\partial V} + P\right)$

Let's now use the fact that $P = a\tau^4$. $\Rightarrow \frac{dP}{d\tau} = 4a\tau^3$.

$$\Rightarrow 4a\tau^3(\tau + d\tau) = \frac{\partial U}{\partial V} + a\tau^4$$

$$\Rightarrow 4a\tau^4 + 4a\tau^3 d\tau = a\tau^4 + \frac{\partial U}{\partial V}$$

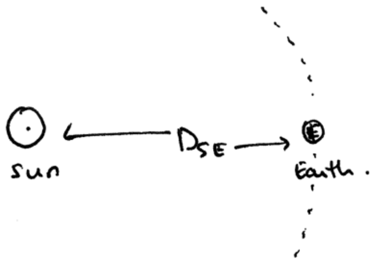
$$\Rightarrow \underbrace{3a\tau^4}_{\text{no differentials}} + \underbrace{4a\tau^3 d\tau}_{\text{a differential}} = \underbrace{\frac{\partial U}{\partial V}}_{\text{a derivative}}.$$

In the limit of small differentials, the second term $\rightarrow 0$.

$$\Rightarrow 3a\tau^4 = \frac{\partial U}{\partial V} \Rightarrow \quad U = 3aV\tau^4$$

(assuming $U \rightarrow 0$ as $V \rightarrow 0$).

4 Surface temperature of the Earth – *Part 1.* Kittel and Kroemer Chp. 4 #5.



Total power radiated by the sun $= J_u \times$ sun's surface area

$$= P_\odot = \sigma_B T_\odot^4 \, 4\pi R_\odot^2 .$$

Total solar power received by earth $= P'_\odot = P_\odot \dfrac{\text{area of earth}}{\text{area of a sphere at } D_{SE}}$

$$\rightarrow \quad P'_\odot = P_\odot \frac{\pi R_E^2 \leftarrow \text{disc}}{4\pi D_{SE}^2 \leftarrow \text{sphere}}$$

Total power radiated by the earth $= J_u \times$ earth's surface area

$$= P_E = \sigma_B T_E^4 \, 4\pi R_E^2 .$$

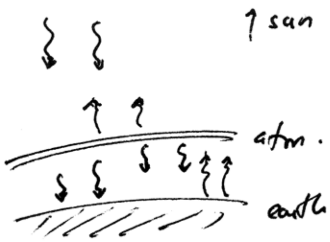
Equil: $\quad P_E = P'_\odot \quad \rightarrow \quad \sigma_B T_\odot^4 \, 4\pi R_\odot^2 \dfrac{\pi R_E^2}{4\pi D_{SE}^2} = \sigma_B T_E^4 \, 4\pi R_E^2$

$$\rightarrow \quad \frac{T_\odot^4 R_\odot^2}{D_{SE}^2} = T_E^4 \, 4 \quad \rightarrow \quad T_E = \left(\frac{T_\odot^4 R_\odot^2}{4 D_{SE}^2} \right)^{1/4}$$

or $\quad \boxed{T_E = T_\odot \left(\dfrac{R_\odot}{2 \, D_{SE}} \right)^{1/2}} \qquad 2\, D_{SE}$

Numerically: $\quad \underline{T_E = 280\ \text{K} = 7°\text{C}}$. a bit cold...

5 Surface temperature of the Earth – *Part 2.* The greenhouse effect.



Earth is in radiative equilibrium w/ atm & sun –

i.e. P_E (see problem 4) $= P'_\odot + P_{atm}\,\frac{1}{2}$

radiated by atmosphere, downward.

Atmosphere is in radiative equil. w/ earth. (absorbs none of sun's radiation).

$$P_{atm} = P_E \,.$$

$$\rightarrow \quad P_E = P'_\odot + \frac{1}{2}P_E \quad \rightarrow \quad P_E = 2P'_\odot$$

$$\rightarrow \quad \sigma_B T_E^4\, 4\pi R_E^2 = 2\,\sigma_B T_\odot^4\, 4\pi R_\odot^2\, \frac{\pi R_E^2}{4\pi D_{SE}^2} \quad \text{(see \# 4)}$$

$$\rightarrow \quad T_E^4 = 2\left(\frac{T_\odot^4 R_\odot^2}{4 D_{SE}^2}\right) \quad \rightarrow \quad T_E = T_\odot\, 2^{1/4}\left(\frac{R_\odot}{2D_{SE}}\right)^{1/2}$$

i.e. the T from #4 times $2^{1/4}$.

Numbers: $T_E = 333K = 60°C$. A bit warm!

6 Kittel and Kroemer Chp. 4 #8.

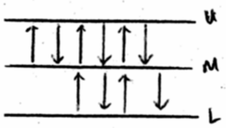


Plate M must be in radiative equilibrium with plates U and L. The radiative flux density hitting a unit area of it (considering both sides) must equal the radiative flux density of its emission (considering both sides).

$J_{UM} + J_{LM}$ **arrows to the middle plate** $= J_{MU} + J_{ML}$ **arrows from the middle plate**

$$\sigma T_U^4 + \sigma T_L^4 = \sigma T_M^4 + \sigma T_M^4 = 2\sigma T_M^4 \Rightarrow \boxed{T_M = \left[\tfrac{1}{2}\left(T_U^4 + T_L^4\right)\right]^{\frac{1}{4}}}$$

Energy flux density without the middle plate $= J_0 = \sigma\left(T_U^4 - T_L^4\right)$

Considering the lower half of the drawing, the energy flux density in the vacuum between plates M and L $= J_{low}$

$$= \sigma\left(T_M^4 - T_L^4\right) = \sigma\left(\tfrac{1}{2}\left(T_U^4 + T_L^4\right) - T_L^4\right) = \sigma\left(\tfrac{1}{2}\right)\left(T_U^4 - T_L^4\right)$$

$$\boxed{J_{low} = \tfrac{1}{2} J_0}$$

Similarly considering the upper half, $J_{up} = \sigma\left(T_U^4 - T_M^4\right)$

$$= \sigma\left(T_U^4 - \tfrac{1}{2}T_U^4 - \tfrac{1}{2}T_L^4\right) = \sigma\left(\tfrac{1}{2}\right)\left(T_U^4 - T_L^4\right) \Rightarrow \boxed{J_{up} = \tfrac{1}{2} J_0}$$

7 Chemical Potential of an Ideal Gas.

$Z_{1,PB} = n_Q V \quad \ldots \quad Z = \frac{(Z_1)^N}{N!} \qquad Z_1 = Z_{1,PB} Z_{int}$

a. $F = -\tau \ln Z \Rightarrow F = -\tau \ln \frac{Z_1^N}{N!} = -\tau \ln Z_1^N + \tau \ln N!$

$\Rightarrow F = -N\tau \ln Z_1 + \tau (N \ln N - N)$ (Stirling)

$\Rightarrow F = -N\tau \ln Z_{1,PB} - N\tau \ln Z_{int} + \tau (N \ln N - N)$.

$\mu = \frac{\partial F}{\partial N}\Big|_{\tau,V}$, so $\mu = -\tau \ln Z_{1,PB} - \tau \ln Z_{int} + \tau \left(N \frac{1}{N} + \ln N - 1\right)$

$\Rightarrow \mu = -\tau \ln(n_Q V) - \tau \ln Z_{int} + \tau \ln N$

$\Rightarrow \mu = \tau \ln\left(\frac{N}{n_Q V Z_{int}}\right)$; $n \equiv \frac{N}{V}$, so

$$\underline{\mu = \tau \ln\left(\frac{n}{n_Q Z_{int}}\right)}$$

b. It is true that $U = \frac{3}{2}N\tau$, and $\mu = \frac{\partial U}{\partial N}\Big|_{V,\sigma}$. However we can't simply evaluate μ by differentiating U since this partial derivative **must** be taken at constant entropy and volume. It is not at all evident from $U = \frac{3}{2}N\tau$ how U would change as a function of N if entropy and volume were held fixed. Entropy for example varies in a complicated way as a function of N, τ, etc. – varying N but keeping entropy fixed, we would certainly have to change τ, which would affect our derivative.