Physics 353: Problem Set 5 - SOLUTIONS

1. Particle states.

a) particles are fermions

One particle in each of the 6 lowest levels; zero in all the others.

b) bosons:

all 6 in the lowest level

C) fermions one possible configuration

For boson system it could be

So there is one possible configuration.
1. + 2 units of energy

Fermions:

\[ \text{empty} \quad \text{or} \quad \text{2 configurations in either case.} \]

Bosons:


\[ f(\varepsilon) = \frac{1}{e^{(\varepsilon - \mu)/kT} + 1} \]

\[ \frac{df}{d\varepsilon} = -\frac{1}{2} \frac{e^{(\varepsilon - \mu)/kT}}{(e^{(\varepsilon - \mu)/kT} + 1)^2} \]

at \( \varepsilon = \mu \)

\[ \left. \frac{df}{d\varepsilon} \right|_{\varepsilon = \mu} = \frac{1}{2} \cdot \frac{e^0}{(e^0 + 1)^2} = \frac{1}{4e} \]

so the lower the temperature, the steeper the slope of the Fermi-Dirac function.


\[ f(\mu + \delta) = \frac{1}{e^{(\mu + \delta - \mu)/kT} + 1} = \frac{1}{e^{\delta/kT} + 1} \]

\[ f(\mu - \delta) = \frac{1}{e^{(\mu - \delta - \mu)/kT} + 1} = \frac{1}{e^{-\delta/kT} + 1} = \frac{e^{\delta}}{e^{\delta/kT} + 1} \]

\[ f(\mu + \delta) + f(\mu - \delta) = \frac{1 + e^{\delta}}{e^{\delta/kT} + 1} = 1 \]

\[ f(\mu + \delta) - f(\mu - \delta) = \frac{1 - e^{\delta}}{e^{\delta/kT} + 1} = 0 \]

so the probability that an orbital \( \delta \) above the Fermi level is occupied is equal to the probability an orbital \( \delta \) below the Fermi level is vacant.
4. Density of orbitals in one and two dimensions.

(a) For electrons in a ONE-DIMENSIONAL box:

\[ E_n = \frac{k^2 n^2}{2m L^2} \text{, where } n \text{ is a positive integer} \]

The Fermi energy \( E_F \) is the energy of the highest filled orbital \((n = \infty)\).

\[ E_F = \frac{k^2 n_f^2}{2m L^2} \text{, where } n_f \text{ is the highest } n. \]

There are \( 2n \) electrons per orbital, so \( N = 2 \times \sum_{n=1}^{n_f} 1 = 2n_f. \)

Therefore, \( E_F \equiv E_F = \frac{k^2 n_f^2 N^2}{8m L^2} \).

We wish to know the DENSITY \( D(\varepsilon) \) of orbitals \( \varepsilon \), that is, \( D(\varepsilon) \) such that \( D(\varepsilon) d\varepsilon \) is the number of orbitals between \( \varepsilon \) and \( \varepsilon + d\varepsilon \).

\[ D(\varepsilon) = \frac{dN}{d\varepsilon} = \frac{N(\varepsilon)}{\varepsilon}. \]

From above, the number \( N \) of free electron orbitals of energy less than or equal to some \( \varepsilon \) is

\[ N(\varepsilon) = \left( \frac{8m L^2}{k^2 \varepsilon} \right)^{1/2}. \]

We could simply differentiate this; here's another way to look at the math:

\[ \ln N = \frac{1}{2} \ln \frac{\varepsilon}{k} + \text{const}. \]

\[ d(\ln N) = d \left( \frac{1}{2} \ln \frac{\varepsilon}{k} + \text{const} \right) \]

\[ \frac{dN}{N} = \frac{1}{2} \frac{d\varepsilon}{\varepsilon}. \]

\[ D(\varepsilon) = \left( \frac{8m L^2}{k^2 \varepsilon} \right)^{1/2}. \]

\[ D(\varepsilon) = \frac{\varepsilon}{\pi} \frac{1}{(2m \varepsilon)^{1/2}}. \]

(b) In two dimensions, \( E_n = \frac{k^2 n^2}{2m L^2} n_r^2 \), where \( n_x^2 + n_y^2 = n_r^2 \). As in part (a), \( E_F = \frac{k^2 n_f^2}{2m L^2} n_r^2 \).

\[ N = 2 \times \sum_{n_r=1}^{n_f} 1 = 2 \times \frac{1}{4} \times \text{Volume of circle of radius } n_f. \]

\[ N(\varepsilon) = \frac{8m L^2}{k^2 \varepsilon} \text{, the number of free electron orbitals less than or equal to some } \varepsilon. \]

\[ \frac{dN}{N} = \frac{d\varepsilon}{\varepsilon}. \]

\[ D(\varepsilon) = \frac{\varepsilon}{\pi L^2}. \]
5, Energy of a relativistic Fermi gas. Kittel & Kroemer #7.2.

\[ E_{n_1,n_2,n_3} = \frac{\hbar c \pi}{L} \left( n_1^2 + n_2^2 + n_3^2 \right)^{1/2} \]
where the “n’s” are integers. The number of orbitals, \( N(\varepsilon) \), with \( \varepsilon < \varepsilon^* \), where \( \varepsilon^* \) is some energy, is the volume of the first octant of a sphere of radius
\[ n_F = \left( n_1^2 + n_2^2 + n_3^2 \right)^{1/2} \]
times 2 for the spin degeneracy. Therefore
\[ N(\varepsilon) = 2 \left( \frac{4}{3} \pi n_F^3 \right) = 2 \frac{1}{3} \pi \left( \frac{L \varepsilon}{\hbar \pi} \right)^3. \]
Noting that \( V = L^3 \), \( N(\varepsilon) = \frac{V}{3 \pi^2 \hbar^3 c^3} \varepsilon^3 \), Therefore the density of states
\[ D(\varepsilon) = \frac{dN}{d\varepsilon} = \frac{V}{\pi^2 \hbar^3 c^3} \varepsilon^2. \]

At \( \tau = 0 \) the occupancy is 1 for \( \varepsilon < \varepsilon_F \) and zero for \( \varepsilon > \varepsilon_F \), allowing us to determine \( \varepsilon_F \) in terms of the particle number, etc.:

\[ N = \int_0^{\varepsilon_F} f(\varepsilon) d\varepsilon \]
\[ = \int_0^{\varepsilon_F} D(\varepsilon) \varepsilon d\varepsilon = \frac{V}{\pi^2 \hbar^3 c^3} \int_0^{\varepsilon_F} \varepsilon^2 d\varepsilon = \frac{1}{3} \frac{V \varepsilon_F^3}{\pi^2 \hbar^3 c^3} \]
\[ \varepsilon_F = \left( \frac{3N \pi^2 \hbar^3 c^3}{V} \right)^{1/3} \]
\[ = \frac{\hbar c}{4} \left( \frac{3N}{\pi} \right)^{1/3}. \]

b) The total energy of the ground state of the gas is
\[ U_0 = \int \varepsilon D(\varepsilon) \varepsilon d\varepsilon = \int_0^{\varepsilon_F} \varepsilon \frac{V}{\pi^2 \hbar^3 c^3} \varepsilon^3 d\varepsilon = \frac{V}{\pi^2 \hbar^3 c^3} \int_0^{\varepsilon_F} \varepsilon^4 d\varepsilon \]
\[ = \frac{V}{\pi^2 \hbar^3 c^3} \frac{3N \pi^2 \hbar^3 c^3}{V} \varepsilon_F^4 \]
\[ \varepsilon_F = \frac{3}{4} N \varepsilon_F. \]