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Physics 353: Problem Set 5 – SOLUTIONS

1. Particle states.

a) particles are fermions

⋮
1
1
1
1
1
1

One particle in each of the 6 lowest levels; zero in all the others.

b) bosons.

⋮
6

all 6 in the lowest level

c) fermions **one possible configuration**

1 ← **1 particle in level 7**
← **empty**
1
1
1
1
1

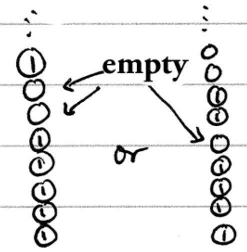
For boson system. it could be

⋮
1
5

So there is one possible configuration.

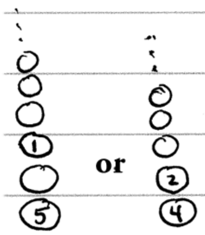
d. + 2 units of energy

Fermions :



2 configurations in either case.

Bosons :



2. **Derivative of the Fermi-Dirac function.** Kittel & Kroemer #6.1.

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/\tau}+1}$$

$$\frac{\partial f}{\partial \varepsilon} = -\frac{\frac{1}{\tau} e^{(\varepsilon-\mu)/\tau}}{\left(e^{(\varepsilon-\mu)/\tau}+1\right)^2}$$

at $\varepsilon = \mu$

$$-\frac{\partial f}{\partial \varepsilon}\Big|_{\varepsilon=\mu} = + \frac{\frac{1}{\tau}\cdot e^{0}}{(e^{0}+1)^2} = \frac{1}{4\tau}$$

So the lower the temperature, the steeper the slope of the Fermi-Dirac function.

3. **Symmetry of filled and vacant orbitals.** Kittel & Kroemer #6.2.

$$f(\mu+\delta) = \frac{1}{e^{(\mu+\delta-\mu)}+1} = \frac{1}{e^{\delta}+1}$$

$$f(\mu-\delta) = \frac{1}{e^{(\mu-\delta-\mu)}+1} = \frac{1}{e^{-\delta}+1} = \frac{e^{\delta}}{e^{\delta}+1}$$

$$\therefore f(\mu+\delta) + f(\mu-\delta) = \frac{1+e^{\delta}}{e^{\delta}+1} = 1$$

$$\therefore f(\mu+\delta) = 1 - f(\mu-\delta)$$

So the probability than an orbital δ above the Fermi level is occupied is equal to the probability an orbital δ below the Fermi level is vacant.

4. **Density of orbitals in one and two dimensions.**

(a) **For electrons in a ONE-DIMENSIONAL box:**

$\varepsilon_n = \frac{\hbar^2\pi^2}{2mL^2} n^2$, where **n is a positive integer**

The Fermi energy ε_F is the energy of the highest filled orbital (at T=0).

$\varepsilon_F = \frac{\hbar^2\pi^2}{2mL^2} n_F^2$, where n_F is the highest n.

There are 2 electrons per orbital, so $N = 2 \times \sum_{n=1}^{n_F} 1 = 2n_F$.

Therefore $\varepsilon_F \equiv \tau_F = \frac{\hbar^2\pi^2 N^2}{8mL^2}$

We wish to know the DENSITY of ORBITALS $D(\varepsilon)$, that is, $D(\varepsilon)$ such that $D(\varepsilon)d\varepsilon$ is the number of orbitals between ε and $\varepsilon + d\varepsilon$. $D(\varepsilon) \equiv \frac{dN}{d\varepsilon}$.

From above, the number N of free electron orbitals of energy less than or equal to some ε is $N(\varepsilon) = \left(\frac{8mL^2\varepsilon}{\hbar^2\pi^2}\right)^{1/2}$.

We could simply differentiate this; here's another way to look at the math:

$\ln N = \frac{1}{2}\ln\varepsilon + \text{const}$. $\quad d(\ln N) = d(\frac{1}{2}\ln\varepsilon + \text{const})$

$\frac{dN}{N} = \frac{1}{2}\frac{d\varepsilon}{\varepsilon}$. $\quad D(\varepsilon) \equiv \frac{dN}{d\varepsilon} = \frac{N(\varepsilon)}{2\varepsilon}$. $\Rightarrow$

$$D(\varepsilon) = \left(\frac{2mL^2}{\hbar^2\pi^2}\right)^{1/2}\frac{1}{\sqrt{\varepsilon}}. \qquad \boxed{D(\varepsilon) = \frac{L}{\pi}\left(\frac{2m}{\hbar^2\varepsilon}\right)^{1/2}}$$

(b) In two dimensions, $\varepsilon_n = \frac{\hbar^2\pi^2}{2mL^2} n^2$, where $n^2 = n_x^2 + n_y^2$. As in part (a), $\varepsilon_F = \frac{\hbar^2\pi^2}{2mL^2} n_F^2$. $\quad N = 2 \times \sum_{n_x}\sum_{n_y}^{(n_x^2+n_y^2 \le n_F^2)} 1 \cong 2 \times \frac{1}{4} \times$ Volume of circle of radius n_F.

$N = \frac{1}{2}\pi n_F^2$. $\quad n_F^2 = \frac{2N}{\pi}$. $\quad \varepsilon_F = \frac{\hbar^2\pi N}{mL^2}$

$N(\varepsilon) = \frac{\varepsilon m L^2}{\hbar^2\pi}$ = the number of free electron orbitals of energy less than or equal to some ε. $\quad \ln N = \ln\varepsilon + \text{const}$. $\quad \frac{dN}{N} = \frac{d\varepsilon}{\varepsilon}$ (see pt. (a))

$D(\varepsilon) \equiv \frac{dN}{d\varepsilon} = \frac{N(\varepsilon)}{\varepsilon}$. $\quad A \equiv L^2$. $\quad \Rightarrow \boxed{D(\varepsilon) = \frac{Am}{\pi\hbar^2}}$

5, Energy of a relativistic Fermi gas. Kittel & Kroemer #7.2.

$E_{n_1,n_2,n_3} = \frac{\hbar c\pi}{L}\left(n_1^2+n_2^2+n_3^2\right)^{1/2}$, where the "n's" are integers. The number of orbitals, $N(\varepsilon)$, with $\varepsilon < \varepsilon^*$, where ε^* is some energy, is the volume of the first octant of a sphere of radius $n_F = \left(n_1^2+n_2^2+n_3^2\right)^{1/2}$ (times 2 for the spin degeneracy). Therefore $N(\varepsilon) = 2\frac{1}{8}\frac{4}{3}\pi n_F^3 = 2\frac{1}{8}\frac{4}{3}\pi\left(\frac{Lc\varepsilon}{\hbar\pi}\right)^3$. Noting that $V = L^3$, $N(\varepsilon) = \frac{V}{3\pi^2\hbar^3c^3}\varepsilon^3$, Therefore the density of states $\boxed{D(\varepsilon) = \frac{dN}{d\varepsilon} = \frac{V}{\pi^2\hbar^3c^3}\varepsilon^2}$.

At $\tau = 0$ the occupancy is 1 for $\varepsilon < \varepsilon_F$ and zero for $\varepsilon > \varepsilon_F$, allowing us to determine ε_F in terms of the particle number, etc.:

$\varepsilon < \varepsilon_F$, $f = 1$

$$\therefore N = \int_0^{\varepsilon_F} f\, D(\varepsilon)\, d\varepsilon$$

$$= \int_0^{\varepsilon_F} D(\varepsilon) d\varepsilon = \frac{V}{\pi^2\hbar^3c^3}\int_0^{\varepsilon_F}\varepsilon^2 d\varepsilon = \frac{1}{3}\frac{V\varepsilon_F^3}{\pi^2\hbar^3c^3}$$

$$\therefore \varepsilon_F = \left(\frac{3N\pi^2\hbar^3c^3}{V}\right)^{\frac{1}{3}}$$

$$= \hbar\pi c\left(\frac{3n}{\pi}\right)^{1/3}.$$

b) the total energy of the ground state of the gas is

$$U_0 = \int f\, D(\varepsilon)\,\varepsilon\, d\varepsilon = \int_0^{\varepsilon_F}\frac{V}{\pi^2\hbar^3c^3}\cdot\varepsilon^3 d\varepsilon = \frac{V}{\pi^2\hbar^3c^3}\cdot\frac{1}{4}\varepsilon_F^4$$

$$= \frac{V}{\pi^2\hbar^3c^3}\cdot\frac{1}{4}\cdot\frac{3N\pi^2\hbar^3c^3}{V}\cdot\varepsilon_F = \frac{3}{4}N\varepsilon_F.$$