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Physics 353: Problem Set 6

Due date: Wednesday, May 21, **5pm**.
Reading: Kittel & Kroemer Chapter 7
Comments: Several problems involve simple numerical **estimates.** Don't consider more than 2 significant figures – we're looking for "order of magnitude" values for comparing various quantities. The problems on this problem set are fairly short except for the last one (#5). For #5, please be sure you understand the Sommerfeld expansion, about which I have distributed a handout.

1, 8 pts. **The sun.** The center of the sun has a density $\rho \approx 10^5$ kg/m^3, and is mostly composed of ionized hydrogen (i.e. protons and electrons). Its temperature is $T \approx 10^7$ K. The mass of a proton is $m_p = 1.7\times10^{-27}$ kg, and the mass of an electron is $m_e = 9.1\times10^{-31}$ kg. (Note $k_B = 1.38\times10^{-23}$ J/K.)

a, 2 pts. Calculate (roughly) the Fermi temperature of protons in the solar interior. Do the protons comprise a degenerate Fermi gas or a classical gas? What about the electrons?

b, 2 pts. The gravitational ("hydrostatic") pressure, P_g, at the center of a sphere of mass M and radius R will depend on M, R, and Newton's gravitational constant $G = 6.67\times10^{-11}$ m^3 / (kg s^2). Using dimensional analysis, derive an expression for P_g in terms of M, R, G, and an unknown dimensionless constant.

c, 1 pt.. Assuming the dimensionless constant ≈ 1, determine P_g at the center of the sun. The mass of the sun is $M_\odot = 2.0\times10^{30}$ kg, and its radius $R = 7.0\times10^8$ m.

d, 1 pt. Calculate the radiation pressure for a blackbody at $T \approx 10^7$ K. (Recall that you derived an expression for the pressure of a photon gas in Problem Set 2.) Is this (roughly) sufficient to balance the pressure you found in part (c)?

e, 2 pt. Calculate the ideal gas pressure or the degenerate Fermi Gas pressure (whatever is appropriate to your answer to part (a)) for the solar interior. Is it (roughly) sufficient to balance the pressure you found in part (c)?

2, 2 pts. **Degeneracy Pressure.** Consider a system of fermions at zero temperature. Using the relation $P = -\left.\frac{\partial U}{\partial V}\right|_{\sigma,N}$, derive an expression for the degeneracy pressure for a system of fermions at $T = 0$. (You may use the relation for U from class.) Show that your relation satisfies the general result we derived last term for non-relativistic particles: $P = \frac{2}{3}\frac{U}{V}$.

3, 3 pts. **Degeneracy Pressure.** Consider a system of fermions at zero temperature.

a, 2 pts. As noted in class, the degeneracy pressure for electrons in a metal is canceled by the electrons' attraction to the positive ions in the metal, and so is not directly observable. A more tangible property is the "bulk modulus" $B = -V\left(\frac{\partial P}{\partial V}\right)_\tau$, which characterizes the resistance of a material to compression. (To generate a fractional change in volume $\frac{\Delta V}{V}$, we need to apply pressure $\frac{1}{B}\Delta P$.) Show that $B = \frac{10}{9}\frac{U}{V}$ at $T = 0$ for degenerate fermions.

b, 1 pt. Imagine that the bulk modulus of copper (density 8000 kg/m³, one conduction electron per atom, electron mass $m_e = 9.1\times10^{-31}$ kg, atomic mass $64\times m_p$, $m_p = 1.7\times10^{-27}$ kg) is solely due to the "$T = 0$" gas of degenerate electrons. Estimate B. Compare to the measured value of about 1.4×10^{11} Pa, and comment on whether the electron gas seems like a significant contributor to the properties of the metal.

4, 8 pts. **A white dwarf star[1].** White dwarf stars are hot and dense and behave as generate electron gases. (There are protons and neutrons present as well, acting as a non-degenerate gas that we can ignore.) The degeneracy pressure of the electrons supports the star. Throughout this problem, assume the temperature $T = 0$. (We'll justify this in part (e).)

a, 2 pts. Show using dimensional analysis that the gravitational potential energy U_g is related to the star's mass M, radius R, and Newton's constant $G = 6.67\times10^{-11}$ m³ / (kg s²) by $U_g = -c\frac{GM^2}{R}$, where c is a positive dimensionless constant. The minus sign does not emerge from dimensional analysis – explain its physical meaning. (*Hint:* which way does the gravitational force point, if we want to remove mass from the star?) By the way: a full analysis yields $c = 3/5$; use this below.

b, 2 pts. Consider there to be one proton and one neutron per electron, and that all particles are nonrelativistic. Show that the energy of the degenerate electrons – i.e. the total kinetic energy of the $T = 0$ "particles in a box" is $U_k = 0.35\frac{\hbar^2 M^{5/3}}{m_e m_p^{5/3} R^2}$, where $m_e = 9.1\times10^{-31}$ kg is the electron mass, $m_p = 1.7\times10^{-27}$ kg is the proton or neutron mass. You can leave the numerical prefactor in terms of π, cube-roots, etc. – I've calculated it to stress that it's not far from 1. *Hint:* There's no new physics in this problem – just think clearly about what the parameters are in our expression for U for fermions, and how to relate these to things like M etc.

c, 2 pts. At equilibrium, we know that the total free energy is minimized; at $T = 0$ the total energy $U = U_g + U_k$ is minimized. Examine U as a function of R, and derive an expression for the equilibrium radius in terms of the star's mass. As the mass increases, does the star's radius increase or decrease? You may find it helpful to sketch $U(R)$.

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[1] based on #7.23 from D. Schroeder, Thermal Physics.

d, 1 pt. Consider a white dwarf star, like Sirius B, whose mass is about the same as our sun's ($M_\odot = 2.0\times10^{30}$ kg). What is its equilibrium radius? How does this compare to our sun's? (See 1c.) How does the density of the star compare to that of water (1 g / cm^3)?

e, 1 pt. Calculate the Fermi temperature for the white dwarf of part d. The temperature of Sirius B is around 25,000 K. Is our $T = 0$ approximation reasonable?

By the way: We're only a few steps away from deriving the "Chandrasekhar limit" – a fundamental upperbound on the mass a star can have and still support itself. In brief: Eventually as M increases we need to consider relativistic degenerate electrons, easy to do given your results from Problem Set 5. It turns out that this "simple" change leads to the consequence that there is **no** possible equilibrium R, and also that there is a maximum M. (The latter fact is a bit harder to prove.) You can read about this if you wish, or wait until one of the end-of-term presentations.

5, 9 pts. $U(\tau)$ **for fermions.** As discussed in class, use the Sommerfeld expansion to derive an expression for the energy of a fermi gas at low temperature. (*Note:* the text takes a different approach to $U(\tau)$. *Don't* do this – use the Sommerfeld expansion. The text's approach is shorter, but it's harder to generalize to new situations.) Note that this derivation proceeds just like the Sommerfeld expansion for N presented in class.

a, 6 pts. First show: $U = \frac{3}{5} N \frac{\mu^{5/2}}{\varepsilon_F^{3/2}} + \frac{3\pi^2}{8} N \frac{\tau\mu^{1/2}}{\varepsilon_F^{3/2}} + \ldots$

b, 3 pts. Then use the expansion for μ derived in class to show: $U = \frac{3}{5} N\varepsilon_F + \frac{\pi^2}{4} N \frac{\tau^2}{\varepsilon_F} + \ldots$