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Physics 353: Problem Set 6 – SOLUTIONS

1. The sun.

(a) $\rho \approx 10^5 \text{ kg/m}^3$. Mostly hydrogen (ionized),
so $m \approx m_{proton} = 1.7 \times 10^{-27}$ kg.

$n = \frac{\rho}{m_p}$ = density of protons and also
= density of electrons (by neutrality)

Fermi energy $\varepsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$.

Fermi Temperature (conventional units) $T_F = \frac{\varepsilon_F}{k_B}$.

$$\rightarrow T_F = \frac{1}{k_B}\frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} = \frac{1}{k_B}\frac{\hbar^2}{2m}\left(3\pi^2 \frac{\rho}{m_p}\right)^{2/3}$$

For the protons, $m = m_p$. Numbers $\rightarrow T_F = 3600$ K (protons)

For the electrons, $m = m_e$. Numbers $\rightarrow T_F = 6.6 \times 10^6$ K

$T \approx 10^7$ K (actual temperature).

Protons : $T \gg T_F$, so the solar protons should not be treated as a degenerate Fermi gas, but rather a classical gas.

electrons : $T \approx T_F$, so the electrons are "intermediate" between behaving as a classical gas and as a degenerate Fermi gas.

b.

Gravitational (hydrostatic) pressure $P = G^a M^b R^c$.

$[P] = \frac{\text{Force}}{\text{area}} = \frac{ML}{T^2L^2}$ (via "F=ma") $= [P] = \frac{M}{LT^2}$

$[G] = \frac{L^3}{MT^2}$. $[R] = L$ $[M] = M$ $\Rightarrow$ $\frac{M}{LT^2} = \left(\frac{L^3}{MT^2}\right)^a M^b L^c$

$\Rightarrow a = 1$ (look at T) $\Rightarrow 3 + c = -1$ (look at L) $\Rightarrow c = -4$

$\Rightarrow -a + b = 1 \Rightarrow b = 2$ (look at M)

$$\Rightarrow \boxed{P = (\#)\, G \frac{M^2}{R^4}}$$

c. Using the given values for M, G, R:

d. From PS2, $P_{blackbody} = \dfrac{\pi^2}{45\hbar^3 c^3}\tau^4$, where $\tau = k_B T$ as usual. Plugging in numbers: $P_{blackbody} = 2.5\times10^{12}$, insufficient to balance P_g: $P_g \approx \dfrac{GM^2}{R^4} = 1.1\times10^{15}$ Pa.

e. For an ideal gas, $P_{idealgas} = n\tau$, so $P_{idealgas} = \dfrac{\rho}{m_p}\tau$ (see part (a)) , independent of whether we consider protons or electrons. Plugging in numbers: $P_{idealgas} = 8.3\times10^{15}$ Pa. This should certainly be relevant for the protons, and we see that it **is** sufficient to balance the gravitational pressure. For $T = 0$ fermions, the degeneracy pressure $P_{FD} = \dfrac{2}{5}\dfrac{N}{V}\varepsilon_F = \dfrac{2}{5}n\varepsilon_F$ (see Problem 2, or the text). For the electrons, plugging in numbers: $P_{FD} = 2.2\times10^{15}$ Pa. The true pressure due to the electrons is therefore somewhere between 2.2×10^{15} and 8.3×10^{15} Pa, in any case sufficient to balance the gravitational pressure.

2. Degeneracy Pressure.

From class, $U_{\tau=0} = \frac{3}{5} N \varepsilon_F$ for 3D degenerate fermions.

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3} .$$

pressure $P = -\left.\dfrac{\partial U}{\partial V}\right|_{\sigma,N}$. $\Rightarrow$ $P_{\tau=0} = -\dfrac{\partial}{\partial V}\left(\dfrac{3}{5}N\dfrac{\hbar^2}{2m}\left(3\pi^2\dfrac{N}{V}\right)^{2/3}\right)$

(note $\sigma =$ const. @ $\tau=0$)

$$\Rightarrow P = +\frac{2}{\cancel{3}}\frac{\cancel{3}}{5} N \frac{\hbar^2}{2m}(3\pi^2)^{2/3} N^{2/3} V^{-5/3}$$

$$\Rightarrow P = \frac{2}{3}\,\frac{3}{5} N \underbrace{\frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}}_{\varepsilon_F =} V^{-1} = \underline{\frac{2}{3}\frac{3}{5}\frac{N\varepsilon_F}{V}} .$$

$$\Rightarrow \underline{P = \frac{2}{3}\frac{U}{V}}$$

3 Degeneracy Pressure.

a

From above, $P = \frac{2}{5} N \frac{\hbar^2}{2m} (3\pi^2 N)^{2/3} V^{-5/3}$.

$$\Rightarrow \frac{\partial P}{\partial V} = -\frac{2}{3} N \frac{\hbar^2}{2m} (3\pi^2 N)^{2/3} V^{-8/3} = -\frac{2}{3} N \varepsilon_F \frac{1}{V^2}$$

$$\Rightarrow B = -V \frac{\partial P}{\partial V} \ @ \tau = 0 \quad = \quad B_{\tau=0} = \frac{2}{3} \frac{N\varepsilon_F}{V}$$

$$= B_{\tau=0} = \frac{2}{3} \frac{5}{3} \frac{U}{V} \quad = \quad \underline{B_{\tau=0} = \frac{10}{9} \frac{U}{V}}$$

b We showed in class that $U = \frac{3}{5} N\varepsilon_F$ and $\varepsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}$, where n is the concentration of electrons. Copper has a density of $\rho = 8000$ kg/m³, with each atom having mass $m = 64 \times m_p$, where $m_p = 1.7 \times 10^{-27}$ kg. Each atom provides one electron, so $n = \rho / m$. Writing also $N = nV$, we can express our relation for B as $B = \frac{10}{9}\frac{U}{V} = \frac{10}{9}\frac{3}{5}\frac{N\varepsilon_F}{V} = \frac{2}{3}\frac{\rho}{m}\frac{\hbar^2}{2m_e}\left(3\pi^2 \frac{\rho}{m}\right)^{2/3} = \frac{1}{3}\frac{\hbar^2}{m_e}\left(3\pi^2\right)^{2/3}\left(\frac{\rho}{m}\right)^{5/3}$, where I've explicitly noted that the mass in the Fermi energy expression is the electron mass. Plugging in numbers in SI units, $B = \frac{\left(1.1 \times 10^{-34}\right)^2}{3 \times 9.1 \times 10^{-31}}\left(3\pi^2\right)^{2/3}\left(\frac{8000}{64 \times 1.7 \times 10^{-27}}\right)^{5/3} = 5.3 \times 10^{10}$ Pa. So the electron gas contributes about a third of the value of the bulk modulus of copper. (The rest is from the crystal lattice.)

4. **A white dwarf star.**

(a) assume $U_g = c' M^{\alpha} R^{\beta} G^{\gamma}$

from dimensional analysis

$$M L^2 T^{-2} = c' M^{\alpha} L^{\beta} L^{3\gamma} M^{-\gamma} T^{-2\gamma}$$

$$= c' M^{\alpha-\gamma} L^{\beta+3\gamma} T^{-2\gamma}$$

$$\therefore \begin{cases} \alpha-\gamma=1 \\ \beta+3\gamma=2 \\ -2\gamma=-2 \end{cases} \Rightarrow \begin{cases} \alpha=2 \\ \beta=-1 \\ \gamma=1 \end{cases}$$

$$\therefore U_g = c' \frac{GM^2}{R}.$$

the gravitational force pointed at the center of the white dwarf star, so if we remove mass from the star, we should do positive work.
so as M decreases, U_g increases,
so there should be a negative sign.

$$\therefore U_g = -c \frac{GM^2}{R}.$$

(b). $M = N(m_e + 2m_p) \approx 2N m_p \quad \therefore N = \frac{M}{2m_p}$

$$\therefore U = \frac{3}{5} N \mu(0)$$

$$= \frac{3}{5} N \cdot \frac{\hbar^2}{2m_e} \left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

$$= \frac{3}{5} \cdot \frac{\hbar^2}{2m_e} \cdot (3\pi^2)^{2/3} \cdot N^{5/3} V^{-\frac{2}{3}}$$

$$= \frac{3}{5} \frac{\hbar^2}{2m_e} (3\pi^2)^{2/3} \cdot \left(\frac{M}{2m_p}\right)^{5/3} \cdot \left(\frac{4}{3}\pi R^3\right)^{-\frac{2}{3}}$$

$$= \frac{3}{5} \cdot \frac{1}{2} \cdot (3\pi^2)^{2/3} \cdot \left(\frac{1}{2}\right)^{5/3} \cdot \left(\frac{4}{3}\pi\right)^{-\frac{2}{3}} \cdot \frac{\hbar^2 M^{5/3}}{m_e m_p^{5/3} R^2}$$

$$= 0.135 \frac{\hbar^2 M^{5/3}}{m_e m_p^{5/3} R^2}$$

(c). $U = U_g + U_k$

$$= -\frac{3}{5}\frac{GM^2}{R} + 0.35\frac{\hbar^2 M^{5/3}}{m_e m_p^{5/3} R^2}.$$

$$\frac{\partial U}{\partial R} = \frac{3}{5}\frac{GM^2}{R^2} - 0.7\frac{\hbar^2 M^{5/3}}{m_e m_p^{5/3} R^3} = 0$$

$\therefore$ $R_0 = \frac{7\hbar^2}{6 m_e m_p^{5/3} G} M^{-\frac{1}{3}}$ is the equilibrium radius.

as mass increases, R_0 decreases.

(d). $M = 2.0 \times 10^{30}\,kg$.

$$R_0 = \frac{7\hbar^2}{6 m_e m_p^{5/3} G} M^{-1/3}$$

$$= \frac{7 \times (1.05 \times 10^{-34}\ J \cdot s)^2}{6 \times 9.1 \times 10^{-31}\,kg \times (1.7 \times 10^{-27}\,kg)^{5/3} \cdot (6.67 \times 10^{-11}\ m^3/(kg\,s^2))} \times (2.0 \times 10^{30}\,kg)^{-1/3}$$

$$\doteq 6.9 \times 10^6\ m.$$

$R_\odot = 7.0 \times 10^8\,m$

$\therefore R_0 \approx \frac{R_\odot}{100}$

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$$

$\therefore \rho_{wds} \approx 10^6 \rho_\odot$

e

2) $$T_F = \frac{\mu(0)}{k_B}$$

$$= \frac{\hbar^2}{2m_e}\left(3\pi^2 n\right)^{2/3}\cdot\frac{1}{k_B}$$

$$n = \frac{N}{V} = \frac{M}{2m_p\,\frac{4}{3}\pi R_0^3} = \frac{3M}{8m_p\pi R_0^3}$$

$$\therefore T_F = \frac{\hbar^2}{2m_e}\left(3\pi^2\,\frac{3M}{8m_p\pi R_0^3}\right)^{2/3}\cdot\frac{1}{k_B}$$

$$= \frac{\hbar^2}{2m_e k_B}\cdot\left(\frac{9\pi M}{8m_p}\right)^{2/3}\cdot\frac{1}{R_0^2}$$

$$= \frac{(1.05\times10^{-34}\,\mathrm{J\cdot s})^2\times(9\times3.14\times2.0\times10^{30}\,\mathrm{kg})^{2/3}}{2\times9.1\times10^{-31}\,\mathrm{kg}\times1.38\times10^{-23}\,\mathrm{J\cdot K^{-1}}\times(8\times1.7\times10^{-27}\,\mathrm{kg})^{2/3}\times(6.9\times10^{6}\,\mathrm{m})^2}$$

$$\doteq 4.2\times10^{9}\ \mathrm{K}.$$

$T_F \gg 25{,}000$ K

So $T=0$ approximation is reasonable.

5. $U(\tau)$ **for fermions.** The derivation closely follows that of μ from the Sommerfeld expansion – see the handout. Start with the relation for U as an integral over the probability distribution:

$$U = \int_0^\infty \varepsilon\, D(\varepsilon)\; f_{FD}(\varepsilon)\, d\varepsilon\,.$$

Continued on the next page.

$D(\varepsilon) = \underbrace{\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}} \varepsilon^{1/2}$ in 3D $= D_0\, \varepsilon^{1/2}$

$D_0 = \qquad\qquad = \frac{3}{2}\frac{N}{\varepsilon_F^{3/2}}$ from earlier ε_F expression

Integrate by parts: $u = f_{FD}$, $dv = \varepsilon^{3/2} d\varepsilon$

$$\Rightarrow \quad U = \underbrace{\frac{2}{5} D_0\, \varepsilon^{5/2} f_{FD}(\varepsilon)\Big|_0^\infty}_{0 \text{ at both lim}} + \frac{2}{5} D_0 \int_0^\infty \underbrace{\varepsilon^{5/2}\left(-\frac{df_{FD}}{d\varepsilon}\right)}_{\text{negligible except near } \mu} d\varepsilon$$

As before, $-\frac{d}{d\varepsilon} f_{FD} = \frac{1}{\tau}\frac{e^x}{(e^x+1)^2}$, where $x = \frac{\varepsilon - \mu}{\tau}$.

$$\Rightarrow \quad U = \frac{2}{5} D_0 \int_0^\infty \frac{1}{\tau}\frac{e^x}{(e^x+1)^2}\varepsilon^{5/2} d\varepsilon = \frac{2}{5} D_0 \int_{-\mu/\tau}^\infty \frac{e^x}{(e^x+1)^2}\varepsilon^{5/2} dx$$

Approx: • lower limit $\to -\infty$ (as in class)

• expand $\varepsilon^{5/2}$ around $\varepsilon = \mu$ ......

$$\varepsilon^{5/2} = \mu^{5/2} + (\varepsilon-\mu)\frac{d}{d\varepsilon}\varepsilon^{5/2}\Big|_{\varepsilon=\mu} + \frac{1}{2}(\varepsilon-\mu)^2\frac{d^2}{d\varepsilon^2}\varepsilon^{5/2}\Big|_{\varepsilon=\mu} + \cdots$$

$$= \mu^{5/2} + \frac{5}{2}(\varepsilon-\mu)\mu^{3/2} + \frac{15}{8}(\varepsilon-\mu)^2\mu^{1/2} + \cdots$$

$$\Rightarrow \quad U = \frac{2}{5} D_0 \int_{-\infty}^\infty \frac{e^x}{(e^x+1)^2}\left[\mu^{5/2} + \frac{5}{2}x\tau\mu^{3/2} + \frac{15}{8}(x\tau)^2\mu^{1/2} + \cdots\right] dx$$

Integrate each term.

Term 1: Deriv. of f_{FD}. (as in class) $\int_{-\infty}^\infty \frac{e^x}{(e^x+1)^2}dx = \int_{\infty}^{\infty} -\frac{df_{FD}}{d\varepsilon}d\varepsilon = f_{FD}\Big|_{\infty}^{-\infty} = 1$

Term 2: as in class, integrand is an odd function of x, so $\int = 0$.

Term 3: As in class, $\int \frac{x^2 e^x}{(e^x+1)^2}dx = \frac{\pi^2}{3}$ **the same integral we saw earlier**

$$\Rightarrow \quad U = \frac{2}{5} D_0 \mu^{5/2} + \frac{2}{5} D_0 \frac{15}{8}\tau^2\mu^{1/2}\frac{\pi^2}{3}$$ Use $D_0 = \frac{3}{2}\frac{N}{\varepsilon_F^{3/2}}$

$$\Rightarrow \quad U = \frac{3}{5} N \frac{\mu^{5/2}}{\varepsilon_F^{3/2}} + \frac{3\pi^2}{8} N\tau^2 \frac{\mu^{1/2}}{\varepsilon_F^{3/2}}$$

Using $\frac{\mu}{\varepsilon_F} = 1 - \frac{\pi^2}{12}\left(\frac{\tau}{\varepsilon_F}\right)^2 + \cdots$ **(see handout)** & keeping terms to $\left(\frac{\tau}{\varepsilon_F}\right)^2$

$$U = \frac{3}{5} N \varepsilon_F \left(\frac{\mu}{\varepsilon_F}\right)^{5/2} + \frac{3\pi^2}{8} N\tau^2 \frac{1}{\varepsilon_F}\left(\frac{\mu}{\varepsilon_F}\right)^{1/2}$$

$$= \frac{3}{5} N\varepsilon_F\left[1 - \frac{5}{2}\frac{\pi^2}{12}\left(\frac{\tau}{\varepsilon_F}\right)^2 + \cdots\right] + \frac{3\pi^2}{8}\frac{N\tau^2}{\varepsilon_F}\left[1 - \frac{1}{2}\frac{\pi^2}{12}\left(\frac{\tau}{\varepsilon_F}\right)^2 + \cdots\right]$$

The product of these two terms is proportional to $\left(\frac{\tau}{\varepsilon_F}\right)^4$ and so is negligible.

$$\Rightarrow U = \frac{3}{5} N\varepsilon_F + N\varepsilon_F\left(\frac{\tau}{\varepsilon_F}\right)^2\left[-\frac{\pi^2}{8} + \frac{3\pi^2}{8}\right] + (\ )\left(\frac{\tau}{\varepsilon_F}\right)^4 \Rightarrow \underline{U = \frac{3}{5}N\varepsilon_F + \frac{\pi^2}{4}N\frac{\tau^2}{\varepsilon_F}}.$$