

## Physics 353: Problem Set 7

**Due date:** Wednesday, May 28, 5pm.

**Reading:** Kittel & Kroemer Chapter 7

**Comments:** This problem set is quite short to help provide time to work on your project. The first two problems require little or no calculation. The last two problems could have been assigned weeks or even months ago – I’m assigning them both to refresh your memory and also in anticipation of their use in our discussion of phase transitions.

**1, 4 pts. General properties of fermion and boson distributions.** Suppose you have a system of fermions in a 3D “box.” Magically, the fermions are transformed into bosons<sup>1</sup>. No property of the particles other than their spin,  $s$ , changes. Which of the following statements are true? Explain.

- A. The density of states,  $D(\epsilon)$ , is unchanged.
- B.  $D(\epsilon)$  retains the same dependence on  $\epsilon$ , but the numerical factor “in front” may change.
- C. The  $\epsilon$ -dependence of  $D(\epsilon)$  changes.
- D. At high temperatures (high compared to any relevant scale) the temperature-dependence of any physical property that does not depend on  $s$  is unchanged.
- E. We don’t have enough information to evaluate statement “D.”
- F. The energy,  $U$ , as  $\tau \rightarrow 0$ , approaches zero. (Define the energy of the lowest level as zero.)

**2, 1 pt. Another general question.** Mr. K. can design any sort of container, of any dimensionality, in which he can store a bunch of electrons – in other words, he can design any density of states function that he chooses, due to his mastery of quantum mechanics. (Assume the function has to be continuous, and isn’t just a “spike” at  $\epsilon = 0$ .) Can he create a system in which the total energy,  $U$ , is independent of temperature? Explain.

**3, 2 pts. Liquid helium.** The density of liquid helium-4 is about 145 kg/m<sup>3</sup>. Considering it as a system of non-interacting bosons, calculate its critical condensation temperature. (Its actual transition from a normal liquid to a zero viscosity “superfluid” occurs at 2.17 K. Interactions between helium atoms are, by the way, very important in determining the behavior of liquid helium.)

**4, 8 pts. Heat capacity of a degenerate Boson gas.** Consider a system of  $N$  non-relativistic spin-0 bosons in a 3D box<sup>2</sup>.

**a, 1 pt.** Write an expression for the energy,  $U$ , of the system. Leave your answer as an integral over  $\epsilon$ .

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<sup>1</sup> This doesn’t actually require magic. Interacting fermions can form attractive pairs; the pairs act as bosons since a pair of half-integer spins has integer spin. This occurs in superconductors and superfluids, and the Bose-Einstein condensation of the resulting system is the essential element of their remarkable properties.

<sup>2</sup> Meaning that there is only one possible spin configuration and so only one orbital per “particle-in-a-box” energy level.

b, 3 pts. For  $\tau < \tau_C$ , the Bose-Einstein condensation temperature, you can set  $\mu = 0$  as discussed in class.

Convert your integral to a dimensionless form, which should be  $\int_0^\infty \frac{x^{3/2}}{e^x - 1} dx$ , which has the value 1.783...

Determine  $U$  as a function of  $\tau$ .

c, 2 pts. Determine an expression for the heat capacity,  $c_V = \left. \frac{\partial U}{\partial \tau} \right|_V$ . Combine symbols and use the relation

for  $\tau_C$  derived in class (or the text) to express your answer solely in terms of  $N$ ,  $\tau$ ,  $\tau_C$ , and a number.

d, 1 pt. Explain why at high temperatures  $c_V \rightarrow \frac{3}{2} N$ .

e, 1 pt. Sketch  $c_V(\tau)$ , using your answer to (c) for the  $\tau < \tau_C$  part and extrapolating to  $\frac{3}{2} N$  for  $\tau > \tau_C$ .

The discrepancy with the text's Figure 7.12 ( $c_V(\tau)$  for liquid helium-4), by the way, arises because of *interactions* between helium atoms, as noted above.

5, 6 pts. **Entropy of mixing.** Consider classical, monatomic ideal gases of the sort we discussed last term, at constant temperature,  $\tau$ . (Note that we derived an expression for the entropy.) Two identical chambers of equal volume,  $V/2$ , are separated by a partition. We'll consider two scenarios.



(Note that part (b) is *different* than part (a) – it's not a continuation of part (a)).

a, 1 pt. Initially, each chamber contains  $N/2$  molecules of the **same** monatomic ideal gas. Call the total entropy of the system  $\sigma_i$ . The partition is removed. Call the new entropy of the system  $\sigma_f$ . Calculate the change in entropy,  $\Delta\sigma = \sigma_f - \sigma_i$ .

b, 2 pts. Initially, each chamber contains  $N/2$  molecules of a **different** monatomic ideal gas – gas  $A$  in chamber 1 and gas  $B$  in chamber 2. Call the total entropy of the system  $\sigma_i$ . The partition is removed. Call the new entropy of the system  $\sigma_f$ . Calculate the change in entropy,  $\Delta\sigma = \sigma_f - \sigma_i$ .

Hopefully you found that the entropy of the system increased. This  $\Delta\sigma$  is called the **entropy of mixing**.

c, 3 pts. Consider a system of two monatomic ideal gases that are not present in equal amounts. The total number of atoms is  $N$ ; a fraction  $x$  are of species  $B$ . Initially each chamber contains a different gas; the concentration in each chamber is the same. Show that the entropy of mixing is given by  $\Delta\sigma_{\text{mixing}} = -N[x \ln x + (1-x) \ln(1-x)]$ . Verify that this gives your answer of part (b) if  $x = 0.5$ .

6, 2 pts. **Free Energy and Work.**

a, 1 pt. From the definition of the Helmholtz Free Energy,  $F$ , and our “original” form of the thermodynamic identity,  $dQ = dU + dW$ , where  $dW$  is the work done **by** the system, show that at constant temperature  $dF = -dW$  -- i.e. the free energy changes by an amount equal to the work done on the system. (We showed this in class long ago.)

b, 1 pt. From the definition of the Gibbs Free Energy,  $G$ , show that at constant pressure  $dG = dF + PdV$  and so at constant temperature and pressure  $dG = -dW + PdV$ . Comment on what this means – is the Gibbs or the Helmholtz free energy a more useful measure of how much “non-mechanical” work you can get from, for example, a chemical reaction?