

Physics 353: Problem Set 7 – SOLUTIONS

1 General properties of fermion and boson distributions.

Changing the particles from fermions to bosons only changes the density of states, $D(\epsilon)$, by a numerical factor corresponding to the spin degeneracy. The rest of $D(\epsilon)$ is determined by the “particle-in-a-box” energy levels, which are unchanged. Therefore statement **B** is true.

The distribution function, f , changes if we change the particle type. However both the Fermi and Bose distributions reduce to the classical (Boltzmann) distribution at high temperature. Therefore statement **D** is true.

For Bosons, as $\tau \rightarrow 0$, all the particles will occupy the lowest, zero-energy orbital. Therefore statement **E** is true.

Statements A and C are false.

2 Another general question.

The distribution function, f , (for Fermions or Bosons) depends on temperature. The density of states **does not**. Since U depends on f via $U = \int f(\epsilon) D(\epsilon) d\epsilon$, this means that no matter what $D(\epsilon)$ Mr. K. creates, U will still depend on temperature.

3 Liquid helium.

The BE condensation temperature $\tau_c = \left(\frac{1}{2.612} \frac{N}{V} \right)^{2/3} \left(\frac{2\pi\hbar^2}{m} \right)$. The number density $n = \frac{N}{V} = \frac{\rho}{m}$,

where ρ is the density, 145 kg/m³. The boson mass $m = 4m_p$, where m_p is the proton mass.

Therefore $\tau_c = 4.3 \times 10^{-23} J$, or more clearly $T_c = 3.1$ Kelvin.

4 Heat capacity of a degenerate Boson gas.

$$U = \langle E \rangle = \int_0^\infty E D(E) f_{BE}(E) dE ; f_{BE} = \frac{1}{\exp(\frac{E-\mu}{\tau}) - 1}$$

Density of states $D(E) = D_0 E^{1/2}$ for ^{non-relativistic} particles in 3D,
 where $D_0 = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2}$ — the same as for
 electrons (spin 1/2) except by a factor of 2 (since our
 spin-0 bosons have only one possible spin configuration).

Recall that the Bose-Einstein temperature T_c is defined by

$$N = 2.612 \left(\frac{m T_c}{2\pi \hbar^2}\right)^{3/2} V, \text{ so}$$

$$D_0 = \frac{1}{4\pi^2} \frac{(4\pi)^{3/2}}{T_c^{3/2}} \frac{1}{2.612} N = \frac{2}{2.612 \sqrt{\pi}} \frac{N}{T_c^{3/2}} = D_0$$

$$(a) \Rightarrow U = \int_0^\infty D_0 \frac{E^{3/2}}{\exp(\frac{E-\mu}{\tau}) - 1} dE. \text{ Define } x = \frac{E-\mu}{\tau}; dE = \tau dx;$$

$$\Rightarrow U = D_0 \tau^{5/2} \int_{-\mu/\tau}^\infty \frac{(x + \mu/\tau)^{3/2}}{e^x - 1} dx$$

(b) For $\tau < T_c$, $\mu \approx 0$.

$$\Rightarrow U \approx D_0 \tau^{5/2} \int_0^\infty \frac{x^{3/2}}{e^x - 1} dx = (1.783) D_0 \tau^{5/2}$$

Using the above expression for D_0 :

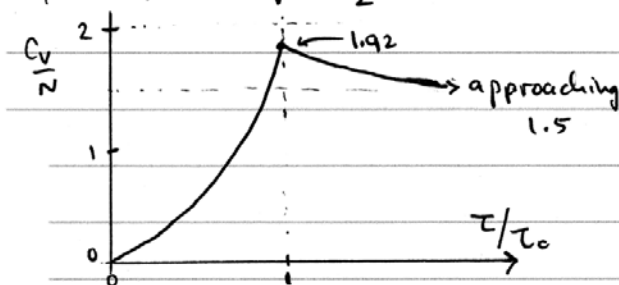
$$U \approx 1.783 \frac{2}{2.612 \sqrt{\pi}} \frac{N}{T_c^{3/2}} \tau^{5/2} = \frac{N}{T_c^{3/2}} \tau^{5/2}$$

$$(c) \text{ Heat capacity } c_v = \frac{\partial U}{\partial \tau} \Big|_V = \frac{5 \cdot 1.783}{\sqrt{\pi} \cdot 2.612} \left(\frac{\tau}{T_c}\right)^{3/2} N$$

$$= c_v = 1.92 \left(\frac{\tau}{T_c}\right)^{3/2} N$$

(d) High τ : $U = \frac{3}{2} N \tau$ (equipartition) $\rightarrow c_v = \frac{3}{2} N$

(e)



5 Entropy of mixing. Classical ideal gas: $\sigma = N \left[\ln(n_Q V / N) + 5/2 \right]$.

(a) Same species in each chamber. Initially, $\sigma_i = \sigma_1 + \sigma_2$, where $\sigma_1 = \sigma_2 = \frac{N}{2} \left[\ln\left(\frac{n_Q V / 2}{N / 2}\right) + \frac{5}{2} \right]$.

Therefore $\sigma_i = N \left[\ln(n_Q V / N) + 5/2 \right]$. Remove partition. $N/2 \rightarrow N$ and $V/2 \rightarrow V$. Final entropy $\sigma_f = N \left[\ln(n_Q V / N) + 5/2 \right]$. We see that $\sigma_f = \sigma_i$, so $\Delta\sigma = 0$.

(b) Different species in each chamber. Initially $\sigma_1 = \sigma_2 = \frac{N}{2} \left[\ln\left(\frac{n_Q V / 2}{N / 2}\right) + \frac{5}{2} \right]$ as above, and so

$\sigma_i = N \left[\ln(n_Q V / N) + 5/2 \right]$. Remove partition. $N/2 \rightarrow N/2$ (unchanged) and $V/2 \rightarrow V$. Final entropy $\sigma_f = \frac{N}{2} \left[\ln\left(\frac{n_Q V}{N / 2}\right) + \frac{5}{2} \right] + \frac{N}{2} \left[\ln\left(\frac{n_Q V}{N / 2}\right) + \frac{5}{2} \right] = N \left[\ln\left(\frac{2n_Q V}{N}\right) + \frac{5}{2} \right]$. Therefore

$\Delta\sigma = N \ln 2$. This is the **entropy of mixing**.

(c) N particles total; $(1-x)N$ are "A", xN are "B." Initially $\sigma_A = (1-x)N \left[\ln\left(\frac{n_Q(1-x)V}{(1-x)N}\right) + \frac{5}{2} \right]$ and

$\sigma_B = xN \left[\ln\left(\frac{n_Q x V}{xN}\right) + \frac{5}{2} \right]$, where we note the partial volume that makes the concentrations of A and B

equal. Therefore $\sigma_i = \sigma_A + \sigma_B = (1-x)N \left[\ln(n_Q V / N) + 5/2 \right] + xN \left[\ln(n_Q V / N) + 5/2 \right]$.

Remove partition. Volumes $\rightarrow V$, with # particles unchanging. Final entropy

$\sigma_f = (1-x)N \left[\ln\left(\frac{n_Q V}{(1-x)N}\right) + \frac{5}{2} \right] + xN \left[\ln\left(\frac{n_Q V}{xN}\right) + \frac{5}{2} \right]$. Therefore

$\Delta\sigma = \sigma_f - \sigma_i = (1-x)N \ln\left((1-x)^{-1}\right) + xN \ln\left(x^{-1}\right)$, canceling out nearly everything in the logarithms.

$\Delta\sigma = -N \left[(1-x) \ln(1-x) + x \ln x \right]$. For $x = 1/2$, $\Delta\sigma = -N \left[-\frac{\ln(2)}{2} + -\frac{\ln(2)}{2} \right] = N \ln 2$, as in (b).

6 Free Energy and Work.

a The Helmholtz Free Energy, $F \equiv U - \tau\sigma$. Therefore $dF = dU - \tau d\sigma - \sigma d\tau$. Via the thermodynamic identity, $dQ = dU + dW$, $\tau d\sigma = dU + dW$, so $dF = -dW - \sigma d\tau$. At constant τ , therefore, $dF = -dW$ -- i.e. the free energy changes by an amount equal to the work done **on** the system.

b The Gibbs free energy $G \equiv U - \tau\sigma + PV = F + PV$, so $dG = dF + d(PV) = dF + PdV + VdP$. At constant pressure, $dG = dF + PdV$. Using the result from (a), at constant temperature and pressure $dG = -dW + PdV$. The change in the Gibbs free energy equals the work done on the system ($-dW$) minus the mechanical work done on the system to expand or contract its volume ($-PdV$). The Gibbs free energy tells us therefore how much work we can get from a system "beyond" that involved in volume changes -- e.g. work we'd use to generate electrical energy, for example.