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University of Oregon; Spring 2008

Physics 353: Problem Set 8

Due date: Wednesday, June 4, **5pm**. No late homework will be accepted.
Reading: Kittel & Kroemer Chapter 10
Comments: **Only part of this problem set will be graded** – you need not turn in the rest.

--------------- GRADED PROBLEMS ---------------

1, 7 pts. **Mean field Ising magnet revisited.** Consider an Ising Magnet in a magnetic field, B, such that the energy of spin i is $E_i = -\lambda B s_i - J s_i \sum_j s_j$. The sum $\sum_j$ is over the nearest neighbors of the spin.

a, 1 pt. Show that for $x \ll 1$, $\tanh x \approx x - x^3/3$. (Recall the properties of hyperbolic functions, e.g. $\frac{d}{dx}\cosh x = \sinh x$, etc., that follow from their definitions, $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$, etc.)

b, 3 pts. Consider $B = 0$. Find an expression for the spontaneous magnetization of the mean-field Ising model, $\langle s \rangle$, when the temperature is close to the critical temperature – in other words, when $\langle s \rangle$ is small. Make use of your answer to (a). You should find that $\langle s \rangle \propto \left(\tau_c - \tau\right)^{\beta}$, where β is a number, known as a **critical exponent.** Determine β. For real 3D magnets, $\beta \approx 1/3$ -- smaller than the value you'll find is predicted by mean field theory.

c, 3 pts. The magnetic susceptibility $\chi = \frac{\partial \langle s \rangle}{\partial B}$. (Actually, χ is defined as $\chi = \frac{\partial M}{\partial B}$, where M includes the constants that relate s to the magnetic moment, but this doesn't change anything important.) Near the critical temperature, derive an expression for χ. You should find that mean-field theory predicts $\chi \propto \left(\tau - \tau_C\right)^{-\gamma}$. Determine γ. Real 3D magnets display this "power law" dependence of χ on temperature, but with $\gamma \approx 1.24$ rather than the value you'll find.

2, 6 pts. **Miscibility transition.** We'll consider a model of a miscibility transition. Consider two components, with an entropy of mixing as calculated in Problem Set 7. The total number of particles is N; xN are species A and $(1-x)N$ are species B. It costs energy to keep A and B next to each other.

a, 1 pt. A simple but reasonable form we might come up with for an interaction energy function is one that is 0 at $x=0$ and $x=1$, that is maximal at $x=1/2$ with a value NU_0, that is symmetric about $x=1/2$, and that is a quadratic function of x. In other words, there's an energetic cost of mixing that's maximal for the most-mixed ($x=1/2$) state, and proportional to the total number of particles. Write the function $U(x)$.

b, 5 pts. Write an expression for the Helmholtz Free Energy, F, as a function of x. As noted in class, the mixture phase-separates if $F(x)$ is concave-down (i.e. its second derivative is negative). For simplicity, just consider the shape of $F(x)$ at $x=1/2$. Determine the critical temperature below which the mixture phase-separates.

3, 4 pts. **Clausius-Clapeyron relation.** The density of ice is 917 kg / m^3. The density of water is 1000 kg/m^3. At 1 atmosphere of pressure, ice melts at 0 °C. The latent heat of melting is 334 kJ/kg.

a, 1 pt. How much pressure must you apply to make ice melt at -1 °C?

b, 1 pt. Consider a glacier at 0 °C. At its base, the weight of the ice presses down; the pressure lowers the melting temperature. Approximately how deep would a glacier have to be for its weight to supply the pressure you calculated in part (a)? (Glaciers can in fact slide on pressure-induced water layers – this mechanism is known as basal sliding.)

c, 2 pts. Some people claim that ice skating works because the pressure under the blade of an ice skate lowers the melting temperature of the ice below the ambient temperature. Perform a numerical estimate and comment on whether this seems plausible.

-------------- UNGRADED PROBLEMS --------------

4 The one-dimensional Ising model. We can exactly solve the one-dimensional Ising model. Consider N spins, each of which can be up ($s=+1$) or down ($s=-1$). The total energy is $E=-J\sum_{ij} s_i s_j$, so $E=-J\left(s_1s_2+s_2s_3+s_3s_4+\ldots+s_{N-1}s_N\right)$. The partition function is therefore $Z=\sum_{s_1=+1,-1}\ \sum_{s_2=+1,-1}\ \ldots \sum_{s_N=+1,-1} \exp\left(\beta J s_1 s_2\right)\exp\left(\beta J s_2 s_3\right)\ldots\exp\left(\beta J s_{N-1}s_N\right)$ where each sum involves two possible values of s, and with $\beta \equiv 1/\tau$.

a. Show that the last sum $\sum_N \exp\left(\beta J s_{N-1}s_N\right)=2\cosh\left(\beta J\right)$, regardless of the value of s_{N-1}.

b. Similarly sum all the other sums, up to $\sum_{s_2}$.

c. Sum the remaining sum over s_1 to show that $Z=2^N\left[\cosh\left(\beta J\right)\right]^{N-1}$. Since $N-1\approx N$ for large N, this gives $Z\approx\left[2\cosh\left(\beta J\right)\right]^N$, with $\beta\equiv 1/\tau$.

d. Calculate U from Z (as we did routinely in 352). Are there any discontinuities or cusps in $U(\tau)$? Hopefully you'll find nothing interesting – no phase transition in one dimension.

5 Free energy of the mean field Ising model. In class we considered the mean-field Ising model from the perspective of the Helmholtz Free Energy. We found that $U(s)$ is a parabolic function of s, and used our earlier expression for the entropy $\sigma(s)$ that is also a parabolic function of s. This yielded a phase transition, but a somewhat nonsensical one, as F was minimized either at $s=0$ or $s=\pm\infty$, not anywhere in between. The problem was $\sigma(s)$ -- our earlier expression was only valid for small s (recall its derivation). Let's take the next step to correct this. Consider $\sigma(s)=N\ln 2-\frac{1}{2}N\langle s\rangle^2-aN\langle s\rangle^4$ -- i.e. throw in a fourth-order term with some coefficient a. Calculate $\langle s\rangle$ as a function of temperature at zero magnetic field. Show that it is zero above a critical temperature and rises smoothly from zero as τ drops below τ_C.