Prof. Raghuveer Parthasarathy
University of Oregon; Spring 2008

Physics 353: Problem Set 8 – SOLUTIONS

1 Mean field Ising magnet revisited.

a.

$$\tanh x = \tanh 0 + \frac{d}{dx}\tanh x\Big|_{x=0} x + \frac{1}{2}\frac{d^2}{d^2x}\tanh x\Big|_{x=0} x^2 + \frac{1}{6}\frac{d^3}{d^3x}\tanh x\Big|_{x=0} x^3 + \ldots$$, Taylor expanding about $x = 0$.

$\frac{d}{dx}\tanh x = \text{sech}^2 x$ (i.e. $\frac{1}{\cosh^2 x}$)

$$\frac{d^2}{d^2x}\tanh x = \frac{d}{dx}\frac{1}{\cosh^2 x} = -2\frac{1}{\cosh^3 x}\sinh x = -2\tanh x\,\text{sech}^2 x$$

$$\frac{d^3}{d^3x}\tanh x = -2\text{sech}^2 x\frac{d}{dx}\tanh x + -2\tanh x\frac{d}{dx}\text{sech}^2 x = -2\text{sech}^4 x + 4\tanh^2 x\,\text{sech}^2 x$$

Noting that $\tanh(0) = 0$ and $\text{sech}(0) = 1$,

$$\tanh x = 0 + 1x + 0x^2 + \frac{1}{6}(-2)x^3 + \ldots$$

$$\tanh x \approx x + \frac{1}{3}x^3.$$

b. From our examination of the mean field Ising model:

$\langle s\rangle = \tanh\left(\frac{JN_n}{\tau}\langle s\rangle\right) = \tanh\left(\frac{\tau_C}{\tau}\langle s\rangle\right)$, using $\tau_C = JN_n$. Near the phase transition, $\langle s\rangle$ is small.

Therefore $\langle s\rangle \approx \left(\frac{\tau_C}{\tau}\langle s\rangle\right) - \frac{1}{3}\left(\frac{\tau_C}{\tau}\langle s\rangle\right)^3$.

$1 \approx \frac{\tau_C}{\tau} - \frac{1}{3}\left(\frac{\tau_C}{\tau}\right)^3\langle s\rangle^2$, so $\langle s\rangle \approx \left[3\left(\frac{\tau}{\tau_C}\right)^3\left(\frac{\tau_C}{\tau} - 1\right)\right]^{1/2}$, and so $\langle s\rangle \approx \left[3\left(\frac{\tau}{\tau_C}\right)^2\left(\frac{\tau_C - \tau}{\tau_C}\right)\right]^{1/2}$. We see that $\boxed{\langle s\rangle \propto (\tau_C - \tau)^{1/2}}$, i.e. the critical exponent $\boxed{\beta = 1/2}$.

(In case you're worried about the other factors of τ, try writing $t \equiv \tau_C - \tau$, with which $\langle s\rangle \approx \sqrt{3}\left(1 - \frac{t}{\tau_C}\right)\left(\frac{t}{\tau_C}\right)^{1/2}$. For τ close to τ_C, the first factor is just something near 1. The second factor changes sharply with t, like $t^{1/2}$.

c. Considering nonzero magnetic field. In the mean field treatment,

$E_i = -\lambda B s_i - J s_i \langle s \rangle = -(\lambda B + J\langle s \rangle) s_i$. Note that the magnetic field just adds a term that "combines" with our coupling factor. Therefore the solution to $\langle s \rangle$ is simply

$\langle s \rangle = \tanh\left(\frac{JN_n \langle s \rangle + \lambda B}{\tau}\right) = \tanh\left(\frac{\tau_C \langle s \rangle + \lambda B}{\tau}\right)$. I'll write two solutions.

Approach 1: Again considering small $\langle s \rangle$ and B near the critical point,

$\langle s \rangle \approx \left((\tau_C \langle s \rangle + \lambda B)/\tau\right) - \frac{1}{3}\left((\tau_C \langle s \rangle + \lambda B)/\tau\right)^3$. Differentiating both sides with respect to B,

$\chi = \frac{d\langle s \rangle}{dB} = \left(\frac{\tau_C}{\tau}\chi + \frac{\lambda}{\tau}\right) - \left(\frac{\tau_C \langle s \rangle + \lambda B}{\tau}\right)^2 \left(\frac{\tau_C}{\tau}\chi + \frac{\lambda}{\tau}\right)$ (Chain rule !)

Evaluating this as $B \to 0$ and $\langle s \rangle \to 0$, $\chi = \frac{\tau_C}{\tau}\chi + \frac{\lambda}{\tau}$, so $\chi = \frac{\lambda}{\tau - \tau_C}$.

We see that $\chi = \frac{d\langle s \rangle}{dB} \propto (\tau - \tau_C)^{-\gamma}$ with $\gamma = 1$.

Approach 2: Avoiding Taylor expansion. From above $\langle s \rangle = \tanh\left((\tau_C \langle s \rangle + \lambda B)/\tau\right)$. Differentiate:

$\chi = \frac{d\langle s \rangle}{dB} = \operatorname{sech}^2\left((\tau_C \langle s \rangle + \lambda B)/\tau\right)\left[\frac{\tau_C}{\tau}\chi + \frac{\lambda}{\tau}\right]$ (Again, Chain rule !). Evaluate as $B \to 0$ and $\langle s \rangle \to 0$, noting that $\operatorname{sech}^2(0) = 1$:

$\chi = 1\left[\frac{\tau_C}{\tau}\chi + \frac{\lambda}{\tau}\right]$, so $\chi\left(1 - \frac{\tau_C}{\tau}\right) = \frac{\lambda}{\tau}$, and therefore $\chi = \frac{\lambda}{\tau - \tau_C}$.

We see that $\chi = \frac{d\langle s \rangle}{dB} \propto (\tau - \tau_C)^{-\gamma}$ with $\gamma = 1$.

2 Miscibility transition.

$U(x) = 0$ @ $x = 0, 1$; $U(x) = U_0 N$ @ $x = 1/2$

parabolic, centered @ $x = \frac{1}{2}$, concave down

$\Rightarrow \quad U(x) = 4U_0 \left[-(x-\frac{1}{2})^2 + \frac{1}{4}\right] N$

$$\underline{U(x) = N U_0 \left(1 - 4(x-\tfrac{1}{2})^2\right)} \; .$$

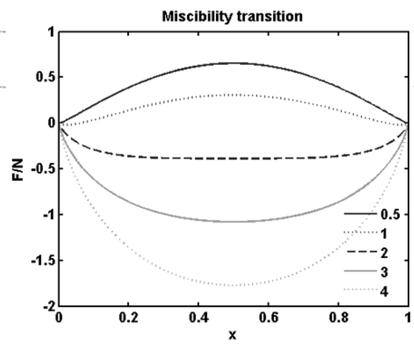
note $x=0 \Rightarrow U=0$, $x=1 \Rightarrow U=0$, $x=\frac{1}{2} \Rightarrow U = U_0 N$

Entropy of mixing $\quad \sigma_{mix} = -N\left[x \ln x + (1-x)\ln(1-x)\right]$

Free energy $\quad F \equiv U - \tau\sigma$

$$F = N U_0 \left[1 - 4(x-\tfrac{1}{2})^2\right] + \tau N\left[x\ln x + (1-x)\ln(1-x)\right]$$

Plot



Phase-separation if $F(x)$ is concave-down, i.e. $\frac{d^2F}{dx^2} < 0$

$$\frac{dF}{dx} = NU_0\left[-8(x-\tfrac{1}{2})\right] + \tau N\left[1 + \ln x + -1 - \ln(1-x)\right]$$

$$= NU_0\left[-8x+4\right] + \tau N\left[\ln\left(\frac{x}{1-x}\right)\right]$$

$$\frac{d^2F}{dx^2} = NU_0(-8) + \tau N\left(\left(\frac{1-x}{x}\right)\left(\frac{+x}{(1-x)^2}\right) + \left(\frac{1}{1-x}\right)\right)$$

$$= -8NU_0 + \tau N\left[\frac{+1}{1-x} + \frac{1}{x}\right]$$

$$= -8NU_0 + \tau N \frac{(1)}{x(1-x)}$$

just consider $x = 1/2$

$$\left.\frac{d^2F}{dx^2}\right|_{x=1/2} = \left.-8NU_0 + \frac{N\tau}{x(1-x)}\right|_{x=1/2} = -8NU_0 + 4N\tau \; .$$

If $\quad -8NU_0 + 4N\tau \begin{cases} < 0, & \text{phase sep.} \\ > 0, & \text{miscible (mixed)} \end{cases}$

$\Rightarrow$ phase sep. for $\quad -2U_0 + \tau < 0 \quad \Rightarrow \quad \boxed{\tau < 2U_0}$.

critical temperature $\quad \underline{\tau_c = 2U_0}$.

3 Clausius-Clapeyron relation.

a. Melting ice at -1 °C. $\frac{dP}{d\tau}=\frac{L}{\tau\Delta v}$, where L is the latent heat per particle and Δv is the volume change per particle. We need to understand all the variables and convert them to appropriate units. Note that dT, one degree, is much smaller than T, 273 K, so we can write that the increase in pressure $dP=\frac{L}{\Delta v}\frac{d\tau}{\tau}$.

A long route: The density of water is 1000 kg/m^3, and each water molecule is $18\times1.67\times10^{-27}$ kg = 3.01×10^{-26} kg, so the volume per water molecule is $3.01\times10^{-26}/1000=3.01\times10^{-29}$ m^3. Similarly the volume per ice molecule is $3.0\times10^{-26}/917=3.28\times10^{-29}$ m^3, so $\Delta v=-0.27\times10^{-29}$ m^3. L = 334 kJ/kg in units of heat per unit mass. Per particle, L is therefore $334\times10^3\times3.01\times10^{-26}=1.01\times10^{-20}$ J. The temperature $\tau=k_B\times273K=3.77\times10^{-21}$ J. Therefore $\frac{dP}{d\tau}=\frac{1.01\times10^{-20}}{3.77\times10^{-21}\left(-0.27\times10^{-29}\right)}=-9.92\times10^{29}\,\frac{\text{Pa}}{\text{J}}$. In conventional temperature units, multiplying by Boltzmann's constant, $\frac{dP}{dT}=-1.37\times10^{7}\,\frac{Pa}{K}$.

More succinctly: write ρ =density, m = mass of a water molecule, so volume per particle $v=\left(\rho/m\right)^{-1}$, i.e. 1/concentration. $\Delta v=\frac{m}{\rho_{water}}-\frac{m}{\rho_{ice}}=m\left(\frac{1}{\rho_{water}}-\frac{1}{\rho_{ice}}\right)=m\left(\frac{\rho_{ice}-\rho_{water}}{\rho_{water}\rho_{ice}}\right)$. The latent heat in units of energy is $L=L'/m$, where L' is the latent heat in the given units of energy / mass. Therefore: $dP=\frac{L}{\Delta v}\frac{d\tau}{\tau}=\frac{dT}{T}\frac{L}{\Delta v}=\frac{dT}{T}\frac{L'\rho_{water}\rho_{ice}}{\left(\rho_{ice}-\rho_{water}\right)}$. Note that the m's and k_B's cancel! Plugging in numbers (SI units): $dP=\frac{-1}{273}\frac{334\times10^3\times917\times1000}{\left(917-1000\right)}=1.35\times10^{7}$ Pa.

Therefore: to lower the melting temperature by one degree, you'd need to apply **13.7 million Pascal** of pressure above normal atmospheric pressure (about 10^5 Pa).

b. Consider a glacier... Using part (a), we need a pressure of 13.7 million Pascal. This pressure is provided by the weight of the ice ρgAh, where ρ is the density and h is the height, divided by its area A. Using ρ from above, we therefore need $\rho gAh/A=\rho gh=1.37\times10^{7}$ Pa/K, so $h=1.53\times10^{3}$ meters, or about **one-and-a-half kilometers.**

c. A 50 kg person skating on an ice skate with a blade with dimensions around 20 cm x 0.1 mm applies a pressure of 2.5×10^{7} Pa. This is certainly enough to lower the melting temperature of ice by 1 degree. Applying $\frac{dP}{dT}=-1.37\times10^{7}\,\frac{Pa}{K}$ from above, it will lower the melting temperature by about 2 °C. However, one can (and usually does) ice skate at temperatures considerably lower than -2 °C. When skating on -10 °C ice it doesn't matter that your weight makes the melting temperature -2 °C – the ice is still solid! Ice skating is not made possible by the Clausius-Clapeyron relation, but rather by heat generated by friction.

4 The one-dimensional Ising model.

a) $\sum_{S_N} e^{\beta J S_{N-1} S_N}$

$= e^{\beta J S_{N-1}} + e^{-\beta J S_{N-1}}$

if $S_{N-1} = 1$, then $\sum_{S_N} e^{\beta J S_{N-1} S_N} = e^{\beta J} + e^{-\beta J} = 2\cosh(\beta J)$

if $S_{N-1} = -1$, then $\sum_{S_N} e^{\beta J S_{N-1} S_N} = e^{-\beta J} + e^{\beta J} = 2\cosh(\beta J)$

(b). $\sum_{S_2=\pm 1} \sum_{S_3=\pm 1} \cdots \sum_{S_N=\pm 1} e^{\beta J S_1 S_2} e^{\beta J S_2 S_3} \cdots e^{\beta J S_{N-1} S_N}$

$= 2\cosh(\beta J) \sum_{S_2=\pm 1} \cdots \sum_{S_{N-1}=\pm 1} e^{\beta J S_1 S_2} \cdots e^{\beta J S_{N-2} S_{N-1}}$

$= \left(2\cosh(\beta J)\right)^2 \sum_{S_2=\pm 1} \cdots \sum_{S_{N-2}=\pm 1} e^{\beta J S_1 S_2} \cdots e^{\beta J S_{N-3} S_{N-2}}$

$= \left(2\cosh(\beta J)\right)^{N-1}$

(c) $\therefore Z = \sum_{S_1=\pm 1} \left(\sum_{S_2=\pm 1} \cdots \sum_{S_N=\pm 1} e^{\beta J S_1 S_2} e^{\beta J S_2 S_3} \cdots e^{\beta J S_{N-1} S_N} \right)$

$= \sum_{S_1=\pm 1} 2^{N-1} \left[\cosh(\beta J)\right]^{N-1}$

$= 2^N \left[\cosh(\beta J)\right]^{N-1}$

for large N, $N-1 \approx N$

$\therefore Z \approx \left[2\cosh(\beta J)\right]^N.$

(d) $U = -\frac{\partial}{\partial \beta} \ln Z = -\frac{\partial}{\partial \beta}\left(N \left(\ln 2 + \ln\cosh(\beta J)\right)\right)$

$= -N \frac{\partial}{\partial \beta}\left[\ln\cosh(\beta J)\right]$

$= +NJ \frac{e^{\beta J} - e^{-\beta J}}{e^{\beta J} + e^{\beta J}} = NJ \tanh(\beta J)$

It's a continuous function. so there is no phase transition in one dimension.

5 Free energy of the mean field Ising model.

From class, $U(\langle s \rangle) = -\frac{N}{2} J N_n \langle s \rangle^2$

$\sigma(s) = N \ln 2 - \frac{N}{2}\langle s \rangle^2 - \alpha N \langle s \rangle^4$, $F \equiv U - \tau\sigma$

$$F(s) = \frac{N}{2}\left\{(-JN_n + \tau)\langle s \rangle^2 - 2\tau \ln 2 + 2\tau\alpha \langle s \rangle^4\right\}$$

$$\frac{\partial F}{\partial s} = \frac{N}{2}\left\{2(-JN_n+\tau)\langle s \rangle + 8\tau\alpha\langle s \rangle^3\right\}$$

$= 0 \Rightarrow \langle s \rangle = 0$ or

$$(-JN_n + \tau) + 4\tau\alpha\langle s \rangle^2 = 0$$

$$\Rightarrow \langle s \rangle^2 = \frac{-\tau + JN_n}{4\tau\alpha}$$

If $\tau > JN_n$, $\langle s \rangle^2 < 0$, not possible so $\langle s \rangle = 0$ is the only solution.

If $\tau \le JN_n$, $\langle s \rangle^2$ has a solution

$\Rightarrow$ $\underline{\tau_c = JN_n}$ (as seen before).

$$\Rightarrow \langle s \rangle^2 = \frac{\tau_c - \tau}{4\alpha\tau} \quad \text{for } \tau \le \tau_c$$

$$\underline{\langle s \rangle = \frac{1}{2\sqrt{\alpha}}\left(\frac{\tau_c}{\tau} - 1\right)^{1/2}} \quad \text{for } \tau \le \tau_c$$

$$\underline{\langle s \rangle = 0} \quad \text{for } \tau > \tau_c$$

Smoothly increasing as τ drops below τ_c.