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Physics 352 – Winter 2008

4 LENSES

We often wish to collect and re-shape electromagnetic wavefronts to create images of objects. **Lenses** are powerful tools for achieving these goals and are obviously very useful, forming the essential imaging elements of telescopes, microscopes, cameras, your eyes, and many other devices. The "ideal" shape of a lens surface is generally some non-spherical conic section (hyperbola, parabola, etc.), but in practice spherical lenses are typically used, since they are vastly easier to make than aspheric (non-spherical) lenses. Typically, one uses spherical lenses and then corrects for their "aberrations" (non-ideal behavior), e.g. by using combinations of lenses.

4.1 A spherical interface

Consider a point source emitting spherical waves from point S, in a medium of index of refraction n_1 (see Figure 4.1). Can we construct a spherical interface of radius R that focuses the emitted light to point P, regardless of where it hits the interface? **What should R be?** Point P is embedded in a medium of index of refraction n_2; we're considering the shape of the interface between medium 1 and medium 2. Consider $n_2 > n_1$, so that the rays from S will be refracted "inwards."

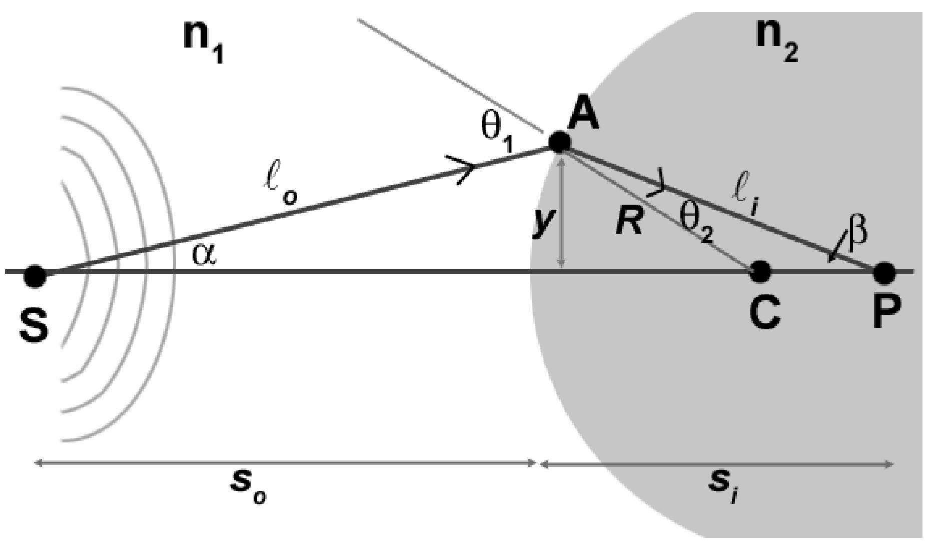

Figure 4.1. A spherical interface. C, S, A,and P refer to particular points – the center of the spherical interface, the object point, the point at which the ray drawn hits the interface, and the image point, respectively. Italicized letters refer to distances. Greek letters refer to angles – note that $\alpha = \angle ASC$ and $\beta = \angle CPA$.

Pay attention to the notation in Figure 4.1. Point C is the center of the sphere of radius R. The distance between the "object" point, S, and the interface is s_o, and the distance between the "image" point, P, and the interface is s_i. The angles that the incident and reflected rays make with respect to the normal to the interface are θ_1 and θ_2. As usual θ_1 and θ_2 are related by Snell's Law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$. We can relate θ_2 to β via the law of sines[2]: $\frac{\sin\beta}{R} = \frac{\sin\theta_2}{s_i - R}$. Relating θ_1 to α isn't quite as transparent; first note that $\angle SAC = \pi - \theta_1$, so $\sin(\angle SAC) = \sin(\pi - \theta_1) = \sin\theta_1$, and then apply the law of sines to $\triangle SAC$ to get: $\frac{\sin\alpha}{R} = \frac{\sin\theta_1}{R + s_o}$. Inserting all this into Snell's Law:

$$n_1 \frac{R+s_o}{R}\sin\alpha = n_2 \frac{s_i - R}{R}\sin\beta \text{, i.e. } n_1 (R+s_o)\sin\alpha = n_2 (s_i - R)\sin\beta .$$

More geometry: $\sin\alpha = \frac{y}{l_o} = \frac{y}{\sqrt{s_o^2 + y^2}} = \frac{y}{s_o\sqrt{1+\left(\frac{y}{s_o}\right)^2}}$ and $\sin\beta = \frac{y}{l_i} = \frac{y}{\sqrt{s_i^2 + y^2}} = \frac{y}{s_i\sqrt{1+\left(\frac{y}{s_i}\right)^2}}$.

Therefore: $$n_1 (R+s_o)\frac{y}{s_o\sqrt{1+\left(\frac{y}{s_o}\right)^2}} = n_2 (s_i - R)\frac{y}{s_i\sqrt{1+\left(\frac{y}{s_i}\right)^2}}$$

$$\boxed{n_1 (R+s_o) s_i\sqrt{1+\left(\frac{y}{s_i}\right)^2} = n_2 (s_i - R) s_o\sqrt{1+\left(\frac{y}{s_o}\right)^2}}$$

We've derived a relation that must hold for focusing at P to occur. In other words, we know what R we need – the R that satisfies the above expression. Unfortunately, it depends on y, the position at which our ray hits the interface! Therefore different rays will not focus to the same image spot.

4.2 A spherical interface – the paraxial regime

What we've shown, in fact, is that a truly spherical interface will not serve as an ideal lens. There's a way out of this, however, which is to limit ourselves to the paraxial regime, meaning that we consider only light that is nearly parallel with the optical axis, SP. In other words, we consider

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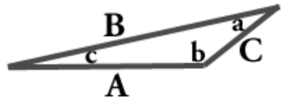
2 Recall from geometry the "Law of Sines:" For any triangle, $\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c}$; see the figure for notation.



small α and β. Therefore, $\frac{y}{s_o}$ and $\frac{y}{s_i}$ are small, allowing us to neglect them in the boxed equation:

$$n_1(R+s_o)s_i \approx n_2(s_i - R)s_o \text{, or}$$

$$n_1 R s_i + n_1 s_o s_i \approx n_2 s_i s_o - n_2 R s_o$$

$$(n_2 - n_1)s_o s_i \approx n_2 R s_o + n_1 R s_i$$

$$(n_2 - n_1)s_o s_i \approx (n_2 s_o + n_1 s_i)R$$

and finally: $\boxed{\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{(n_2 - n_1)}{R}}$. A simple, clean, useful relation! See Figure 4.2 for an illustration of the paths of many different rays.

(By the way, we could also have derived this directly from Fermat's principle, by determining the R for which *SAP* is an extremal path for any *A*.)

Should we be bothered by limiting ourselves to the paraxial case? Yes and no. In practice one **does** try to design optical systems such that beams are close to the center of spherical lens elements or, equivalently, to have one's image and object distances be large compared to the size of the lens. If one does this, the above relation words very well. In practice, one works in the paraxial regime, and applies additional corrections if necessary.

We will continue thinking about the paraxial regime.

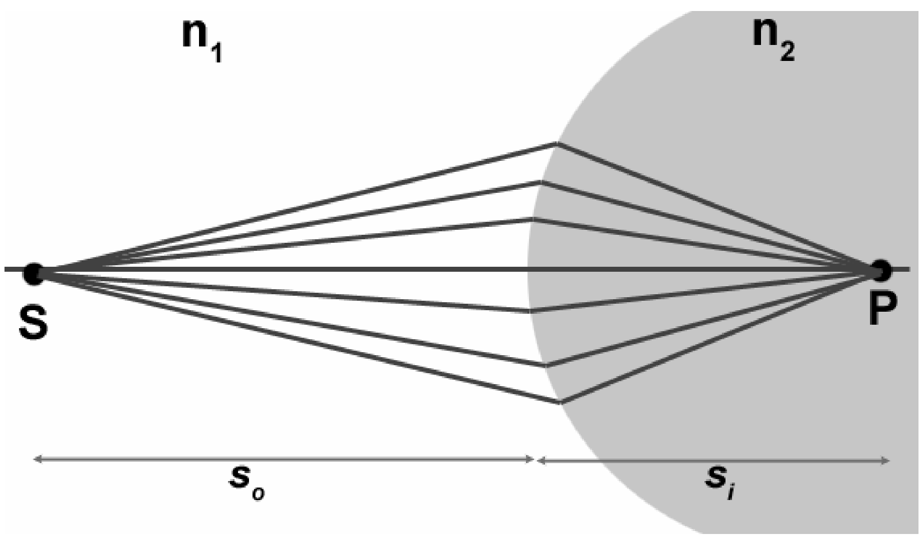


Figure 4.2. Focusing by a spherical interface – the paraxial regime, in which all rays from S are refracted to P.

4.3 Focal Points

If R, n_1 and n_2 are fixed, decreasing s_o means that s_i increases (and vice versa), from the above boxed relation. Let's increase s_i until $s_i \to \infty$, in other words parallel rays emerge from the interface; what is s_o? From above: $\frac{n_1}{s_o} + \frac{n_2}{\infty} = \frac{(n_2 - n_1)}{R}$, therefore $s_o = \frac{n_1}{(n_2 - n_1)} R$ -- an object at this distance focuses "to infinity." We'll call this distance the **object focal length**, $f_o \equiv \frac{n_1}{n_2 - n_1} R$. The spherical waves from the point source turn into plane waves – see Figure 4.3.

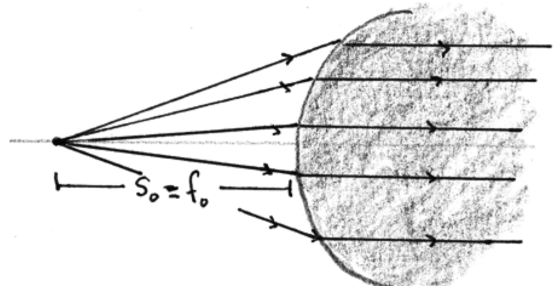


Figure 4.3. Light emanating from the object focal distance is focused to an image distance of infinity (i.e. rays become parallel).

The same holds if we don't consider a "semi-infinite" medium on the right, but rather a finite lens with a spherical surface at the left and a flat surface at the right – a **plano-convex lens** (See Figure 4.4.) Note that since the right edge is flat, all rays are normal to it, and there is no "bending" of the rays due to refraction. Plano-convex lenses are very useful.

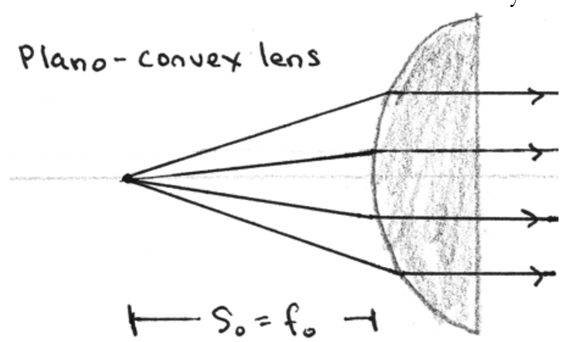


Figure 4.4. Parallel rays generated by a plano-convex lens from a source located at the object focal length.

We can of course consider the opposite situation, in which plane waves (parallel rays from $s_o = \infty$) are focused to an image at some s_i. This particular s_i is denoted f_i, the **image focal length**.

4.4 Real and Virtual Images

Solving the boxed lens equation in Section 4.2 for s_i, we have $s_i = n_2 \left(\frac{n_2 - n_1}{R} - \frac{n_1}{s_o} \right)^{-1} = \frac{n_2}{n_1} \left(\frac{1}{f_o} - \frac{1}{s_o} \right)^{-1}$. If $s_o > f_o$, then $s_i > 0$, and point P is to the right of the interface. The rays from S converge at P. To an observer at the right, *it looks as if light is* *emanating from point* P. We have what's called a "**real image**" at P (see Figure 4.5). If, for example, we put a power meter at P, we detect a high degree of power due to the focused light

If $s_o < f_o$, then $s_i < 0$, and point P is to the left of the interface. The rays don't actually hit point P, but they *appear* to an observer at the right as if they are emanating from P (see Figure 4.5). We have what's called a "**virtual image**" at P. If, for example, we put a power meter at P, we **do not** detect a high intensity focused spot, since there is no "spot" there.

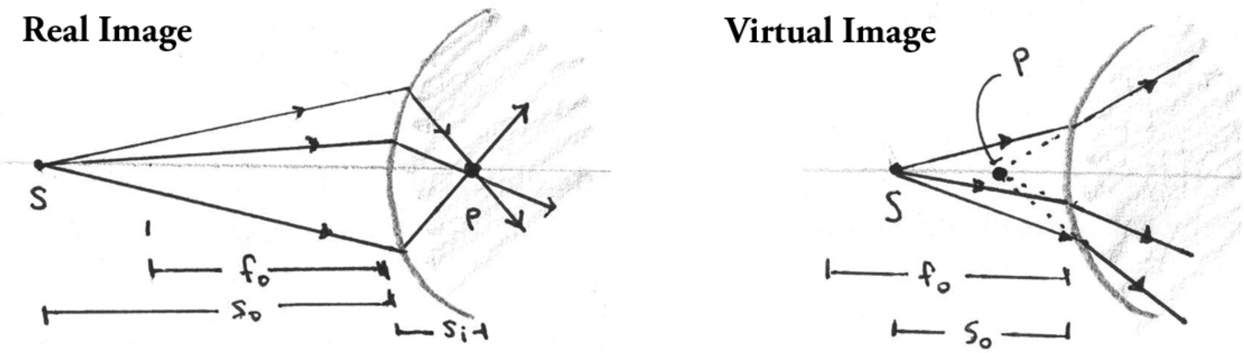


Figure 4.5. Real and virtual images. Left: light emanates from *P*. Right: Light looks to an observer like it's emanating from point *P* located to the left of the interface.

4.5 Concave lenses

The same analysis works for concave lenses, but we treat R **as negative** ($R < 0$). Since $\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}$, if $n_2 > n_1$ then $s_i < 0$ – we have a virtual image (see Figure 4.6).

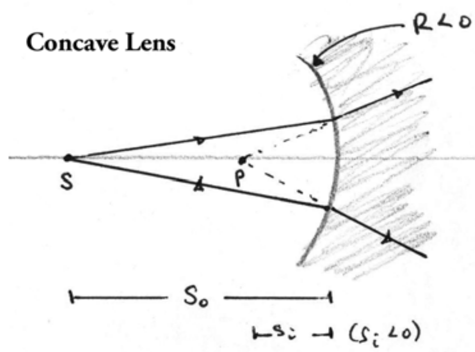


Figure 4.6. A convex lens. Note the virtual image (if $n_2 > n_1$).

4.6 Thin lenses

Let's glue one lens of radius of curvature R_1 onto another of R_2. (See Figure 4.7.) We'll consider thin lenses, and so neglect the lens thickness d (i.e. we're assuming d is smaller than other lengths involved).

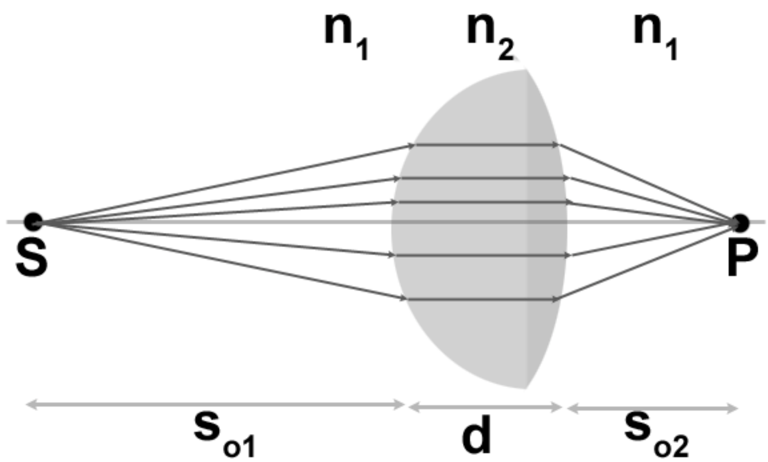


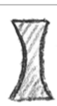
Figure 4.7. Focusing light with a thin lens (imagine d is small).

The object and image lengths for "lens 1" (the left half of the lens) are related by

$\frac{n_1}{s_{o1}} + \frac{n_2}{s_{i1}} = \frac{n_2 - n_1}{R_1}$. The image of lens 1 provides the "object" for lens 2. Therefore $s_{02} = -s_{i1} + d \approx -s_{i1}$, where the negative sign arises because, as defined above, a positive image length and a positive object length lie in opposite directions. Considering lens 2:

$\frac{n_2}{-s_{i1}} + \frac{n_1}{s_{i2}} = \frac{n_1 - n_2}{R_2}$, where we keep track of which index of refraction is which.

We need to adopt a **consistent set of sign conventions** for the radii. As noted above, a convex "left" lens has $R > 0$, and a concave "left" lens has $R < 0$. For the right side lens, these are switched. Here are some illustrations of these rules:

	Biconvex, R_1>0, R_2<0
	Planar convex, R_1=∞, R_2<0
	Planar convex, R_1>0∞, R_2=∞
	Meniscus convex, R_1>0, R_2>0
	Biconcave, R_1<0, R_2>0

Returning to our thin lens, adding the two expressions above:

$$\frac{n_1}{s_{o1}} + \frac{n_1}{s_{i2}} = (n_2 - n_1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

For a **thin lens in air**, $n_1 \approx 1$; $n_2 = n_{lens}$, giving us the **Thin Lens Equation**, or **Lensmaker's Formula**:

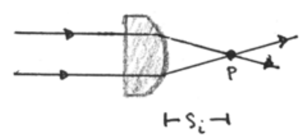
$$\frac{1}{s_o} + \frac{1}{s_i} = (n_{lens} - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

The **focal length**, f, is given either by s_o or $s_i \to \infty$ (it doesn't matter which):

$$\frac{1}{f} = (n_{lens} - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$ We can then write the thin lens equation as

$$\boxed{\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}}$$, also known as the Gaussian Lens Formula. This is one the most important relations for the design of optical systems.

For example: Consider parallel rays incident on a glass ($n = 1.5$), $R_1 = \infty$, $R_2 = -50$ mm plano-convex lens. (See the figure, right, and note the relation between the shape of the lens surfaces and the signs of the R's.) Where will these rays be focused to? *Answer:*



$$\frac{1}{f} = (1.5 - 1)\frac{1}{50mm}$$, so f = 100 mm, s_i = 100 mm.

4.7 Magnification

Lenses magnify objects. The magnification can be >1 or <1 (which Hecht calls "minification"). See the figure of the thin lens, below, which is magnifying an extended object (i.e. not a point source) – in this case, an apple.

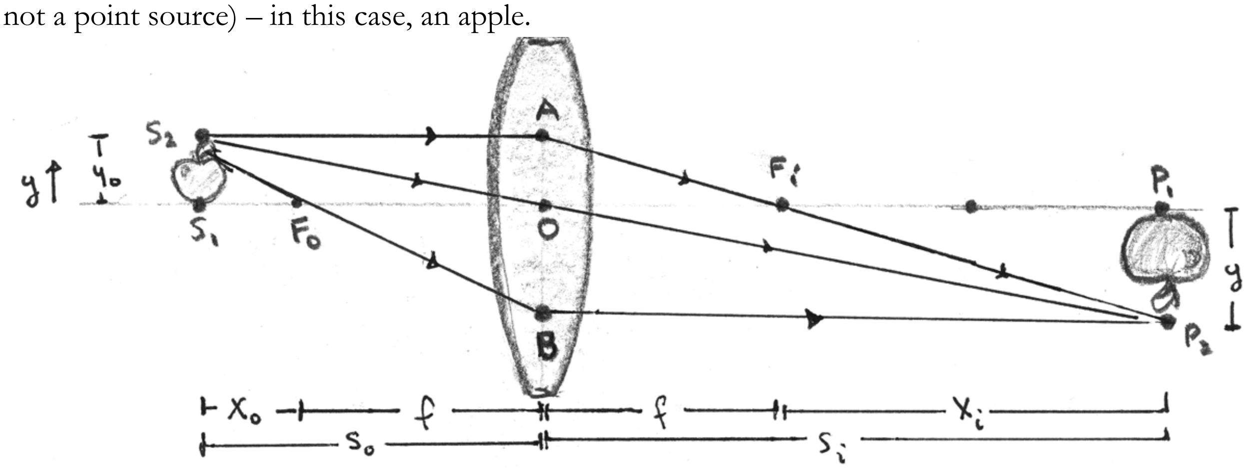


Figure 4.8. Lenses magnify images – schematic.

Points F_o and F_i are each a distance f, the focal length, from the lens. Consider light emanating from the top of the apple. The ray that goes through F_o will emerge from the lens parallel to the axis (think: *why*?). The ray the leaves the apple parallel to the axis will go through F_i (think: *why*?). The ray that goes through the center of the lens will be undeflected in the thin lens limit – see Hecht for a discussion.

The **magnification**, M_T, is defined to be the height of the image relative to the height of the object – i.e. $M_T \equiv y_i / y_o$. Triangle S_1S_2O is similar to triangle P_1P_2O, so $y_o / s_o = -y_i / s_i$, so $M_T \equiv -s_i / s_o$; the negative sign shows that the image is **inverted**.

Triangle AOF_i is similar to triangle $P_1P_2F_i$, so $y_o / f = |y_i| / (s_i - f)$.

Triangle BOF_o is similar to triangle $S_1S_2F_o$, so $y_o / (s_o - f) = |y_i| / f$. Combining these, $\frac{f}{s_o - f} = \frac{s_i - f}{f}$. Note that $s_i - f = x_i$ (see figure). Using the last similar triangle relation again, $\frac{f}{s_o - f} = \frac{-y_i}{y_o}$. And so: $\boxed{M_T = y_i / y_o = -\frac{x_i}{f}}$. We could also have written: $\boxed{M_T = -\frac{f}{x_o}}$.

As the object distance x_o is lowered, the magnification increases.

(*Think about what happens if $x_o < 0$, i.e. the object is closer than the focal point. Drawing rays, you should be able to convince yourself that the lens cannot form an image of the object.*)

4.8 Resolution

When considering single-slit diffraction, we realized that $\theta_{min} \approx \lambda/a$, where λ is the wavelength of light and a is the diameter of the imaging device, is the angular resolution of the device. Two objects must have an angular separation of at least θ_{min} if they are to be resolved as separate objects. Using lenses to magnify objects, this angular resolution criterion still holds. Moreover, the fact that the object distance can't be closer than the focal length turns our resolution relation into a distance criterion, as we'll (probably) see in Problem Set 4.

To be resolvable by a magnifying lens, two objects must be separated by at least $x_{min} \approx \lambda$. As before, we're ignoring factors of 2, etc., in this expression. The "true" expression, if you're interested, is $\mathbf{x_{min} = \lambda/(2\, n \sin\theta_m)}$, where n is the index of refraction of the medium and θ_m is the maximal angle of the "cone" of light the lens collects. For optical wavelengths ($\lambda \approx 500$ nm), imaging in water (n=1.3), this expression sets a fundamental limit of at best $\approx$ 200 nm for the resolution of microscopes. As we'll see later in class, many people are working on clever tricks to get around this "**diffraction limit**."