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5 Reflection and Transmission (Fresnel's Equations)

The law of reflection ($\theta_r = \theta_i$, where r and i refer to reflected and incident rays – see Problem Set 3) and Snell's Law ($n_i \sin\theta_i = n_t \sin\theta_t$, where t refers to the transmitted ray) give the **directions** of reflected and transmitted rays at boundaries (Figure 5.1). What are the **amplitudes** of the electromagnetic waves? In other words, **how much** light is reflected and transmitted?

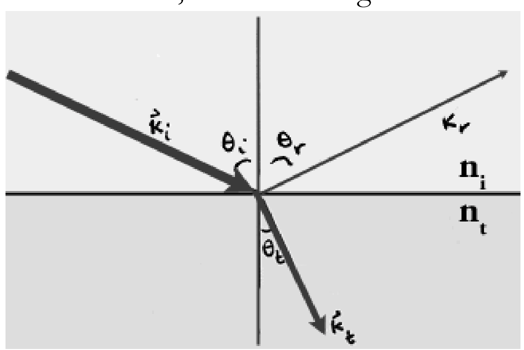

Figure 5.1. Reflection and refraction at an interface. The incident wave (wavevector $\vec{k}_i$) is reflected (wavevector $\vec{k}_r$) and transmitted (wavevector $\vec{k}_t$). What are the **amplitudes** of the reflected and transmitted waves?

Similar questions arise when considering other sorts of waves hitting boundaries – for example waves on the 1D strings we examined last quarter, incident at a boundary between two media with different speeds. In all these situations, transmission and reflection are analyzed by considering the **boundary conditions** imposed by the junction. As we'll see, our analysis of electromagnetic waves will reduce in the appropriate limit to that of simple strings.

To consider the general case of a plane electromagnetic wave hitting a surface at some angle θ_i (with respect to the normal), we'll have to separately consider the components with electric field **perpendicular** and **parallel** to the **plane of incidence**. (The incident, reflected, and transmitted rays all lie in the plane of incidence, POI, which also includes the normal to the surface.)

First, some statements that follow from a study of electromagnetism. (We won't derive these.)

- The electric and magnetic field vectors of an EM wave are perpendicular to each other.
- $\vec{E} \times \vec{B}$ points along $\vec{k}$, the wavevector, i.e. along the direction of propagation.
- The field amplitudes are related by $|\vec{E}| = v|\vec{B}|$, where $v = c/n$ is the wave speed.

The **boundary conditions** at the interface between media are:

(*i*) The tangential (i.e. parallel to the interface) components of the electric field, $\vec{E}$ are continuous across the boundary.

(*ii*) The tangential (i.e. parallel to the interface) components of $\vec{B}/\mu$, where μ is the magnetic permeability of the medium, are continuous across the boundary.

Let's consider the two cases.

5.1 Case I: $\vec{E}$ perpendicular to the plane of incidence

See Figure 5.2. (Note that a circle with a dot in it indicates a vector, in this case an electric field vector, that points out of the page towards you.)

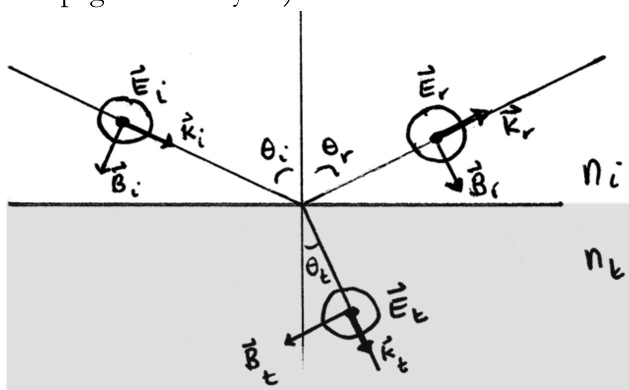

Figure 5.2. Electric and magnetic field vectors for light polarized with $\vec{E}$ perpendicular to the plane of incidence.

The electric field vectors are completely tangential to the interface. The magnetic field vectors are not. Applying boundary condition (i) to the amplitudes (E_0) of the electric fields:

$$E_{0i} + E_{0r} = E_{0t}$$

Applying boundary condition (ii) to the amplitudes (B_0) of the magnetic fields:

$$-\frac{B_{0i}}{\mu_i}\cos\theta_i + \frac{B_{0r}}{\mu_i}\cos\theta_r = -\frac{B_{0t}}{\mu_t}\cos\theta_t$$; see the figure to understand the signs.

Using $B_0=E_0/v$ (from above), $v_i=v_r$ (since they are in the same media), $\theta_i=\theta_r$ (law of reflection), and $v_i=c/n_i$, we can write the above relation as:

$$\frac{n_i}{\mu_i}\left(E_{0i} - E_{0r}\right)\cos\theta_i = \frac{n_t}{\mu_t}E_{0t}\cos\theta_t .$$

Combining this with the boundary condition (i) equation above, substituting to eliminate E_{0t} (some dull algebra worked through in class), we can solve for **the ratio of the reflected wave amplitude to the incident wave amplitude**:

$$\left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{\frac{n_i}{\mu_i}\cos\theta_i - \frac{n_t}{\mu_t}\cos\theta_t}{\frac{n_i}{\mu_i}\cos\theta_i + \frac{n_t}{\mu_t}\cos\theta_t}.$$

Similarly solving instead for **the ratio of the transmitted wave amplitude to the incident wave amplitude**:

$$\left(\frac{E_{0t}}{E_{0i}}\right)_\perp = \frac{2\frac{n_i}{\mu_i}\cos\theta_i}{\frac{n_i}{\mu_i}\cos\theta_i + \frac{n_t}{\mu_t}\cos\theta_t}.$$

Typically, one deals with nonmagnetic materials: $\mu \approx \mu_0$, the permeability of free space. The above equations simplify, yielding two of the four **Fresnel Equations**, for the amplitude reflection coefficient $r_\perp$ and the amplitude transmission coefficient $t_\perp$.

$$r_\perp = \left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$t_\perp = \left(\frac{E_{0t}}{E_{0i}}\right)_\perp = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t}.$$

5.2 Case II: $\vec{E}$ parallel to the plane of incidence

See Figure 5.3. (Note that a circle with a dot in it indicates a vector that points out of the page towards you.)

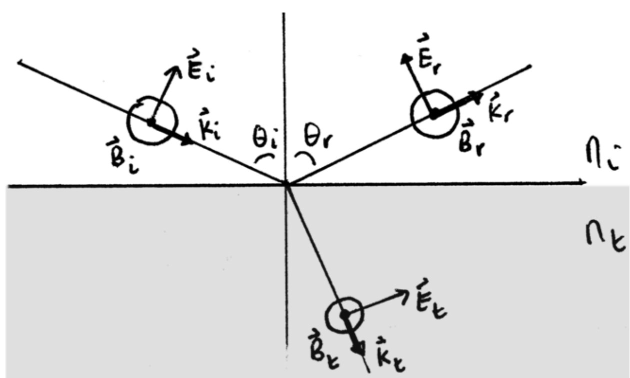

Figure 5.3. Electric and magnetic field vectors for light polarized with $\vec{E}$ parallel to the plane of incidence.

Applying the boundary conditions to this geometry (which you'll do in the homework) leads to:

$$\left(\frac{E_{0r}}{E_{0i}}\right)_{//} = \frac{\frac{n_t}{\mu_t}\cos\theta_i - \frac{n_i}{\mu_i}\cos\theta_t}{\frac{n_i}{\mu_i}\cos\theta_i + \frac{n_t}{\mu_t}\cos\theta_t} \text{ and}$$

$$\left(\frac{E_{0t}}{E_{0i}}\right)_\perp = \frac{2\frac{n_i}{\mu_i}\cos\theta_i}{\frac{n_i}{\mu_i}\cos\theta_t + \frac{n_t}{\mu_t}\cos\theta_i}.$$

For typical nonmagnetic media we get the other two **Fresnel Equations**:

$$r_{//} = \left(\frac{E_{0r}}{E_{0i}}\right)_{//} = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_i\cos\theta_t + n_t\cos\theta_i}$$

$$t_{//} = \left(\frac{E_{0t}}{E_{0i}}\right)_{//} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i}.$$

5.3 Brewster's Angle

Let's plot $r_\perp$ and $r_\parallel$ as a function of θ_i for light incident from air ($n_1 = 1$) to water ($n_t = 1.33$) – see Figure 5.4.

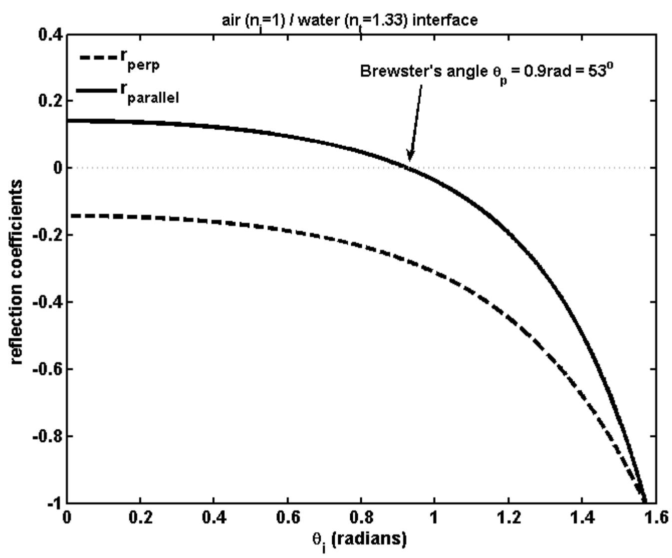

Figure 5.4. $r_\perp$ and $r_\parallel$ for light incident from air to water, as a function of incidence angle. Note that $r_\parallel$ crosses zero!

We see something very interesting: there is some particular θ_i for which the reflection coefficient is **zero** for light with its electric field parallel the plane of incidence. There is no such angle for the perpendicular polarization.

We could examine this more carefully and show, after lots of lines of algebra, that if $n_t > n_i$, $r_{\parallel} = 0$ at one particular incident angle, θ_i. This angle is called **Brewster's Angle,** θ_p, and is given by $\tan\theta_p = n_t / n_i$. All the parallel-polarized light is transmitted.

What about the perpendicular polarization? Again, we could show that there is **no** angle that gives $r_{\perp} = 0$.

Therefore, shining randomly polarized light incident at the Brewster angle, the reflected light is completely polarized with its electric field perpendicular to the plane of incidence.

As I'll discuss at some point in class, knowing the mysteries of the Brewster's Angle has saved me from difficult real-life dilemmas!

5.4 Waves on Strings at Boundaries

How does this relate to wave propagation in a 1D string? We could quite simply derive reflection and transmission amplitudes by continuing our analysis of last quarter[3]. Let's instead use the Fresnel equation analysis above. For the 1D string, there is, by definition, only one dimension! So only $\theta_i = 0$ is meaningful, for which $\theta_t = 0$ (Snell's Law). The reflection amplitude $r_{\perp} = -r_{\parallel} = \frac{n_i - n_t}{n_t + n_i} = \frac{\frac{c}{v_i} - \frac{c}{v_t}}{\frac{c}{v_t} + \frac{c}{v_i}} = \frac{v_t - v_i}{v_t + v_i}$. In other words, the amplitude of the reflected wave is proportional to the **difference** in velocities of waves in the two media. (In case you're worried about why $r_{\perp}$ and $r_{\parallel}$ have opposite signs, see Hecht section 4.6.3.)

Numbers: For normal incidence ($\theta_i = 0$, $\theta_t = 0$) at an air (n_i =1) / glass (n_t = 1.5) interface, the reflection amplitude $r = 0.2$.

[3] See, for example, Chapter 8 of A. P. French, *Vibrations and Waves*.