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Problem Set 1: SOLUTIONS

(1, 5 pts.**) Spherical Waves.**

(a) $\psi(\vec{r},t) = \frac{A}{r}\exp[j(kr-\omega t)]$

$$\therefore \nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$= \frac{\partial}{\partial r}\left(-\frac{A}{r^2}e^{j(kr-\omega t)} + \frac{A}{r}(jk)e^{j(kr-\omega t)}\right)$$

$$+ \frac{2}{r}\left(-\frac{A}{r^2}e^{j(kr-\omega t)} + \frac{A}{r}(jk)e^{j(kr-\omega t)}\right)$$

$$= \frac{2A}{r^3}\cancel{e^{j(kr-\omega t)}} - \cancel{\frac{A}{r^2}(jk)}e^{j(kr-\omega t)} - \cancel{\frac{A}{r^2}(jk)}e^{j(kr-\omega t)}$$

$$+ \frac{A}{r}(-k^2)e^{j(kr-\omega t)} - \cancel{\frac{2A}{r^3}e^{j(kr-\omega t)}} + \cancel{\frac{2A}{r^2}(jk)}e^{j(kr-\omega t)}$$

$$= -\frac{A}{r}k^2 e^{j(kr-\omega t)}$$

$$\frac{\partial^2\psi}{\partial t^2} = \frac{\partial}{\partial t}\left(-j\omega\frac{A}{r}\exp[j(kr-\omega t)]\right)$$

$$= -\omega^2\frac{A}{r}\exp(j(kr-\omega t))$$

$$v = \frac{\omega}{k}$$

$$\therefore \frac{1}{v^2} = \frac{k^2}{\omega^2}$$

$$\therefore \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = -\frac{k^2}{\omega^2}\cdot\omega^2\frac{A}{r}\exp[j(kr-\omega t)]$$

$$= -\frac{A}{r}k^2 e^{j(kr-\omega t)}$$

$$= \nabla^2\psi$$

$\therefore$ ψ is a solution to the wave equation $\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$.

(b) How are power and intensity related? Intensity = Power / Area. What is the intensity of the wave? $I \propto \vec{E} \cdot \vec{E}^*$, and so $I \propto \frac{|A^2|}{r^2} e^{j(kr-\omega t)} e^{-j(kr-\omega t)} = \frac{|A^2|}{r^2}$. The total power incident on the sphere is therefore $P = IA = I 4\pi R^2 \propto \frac{|A^2|}{R^2} 4\pi R^2$. The "R"s cancel, and we see that the total power is independent of R! This makes sense: The power isn't "lost" as the wave propagates from the point – think about energy conservation – and so the total power incident on any shell should be the same.

2. Wave sizes and speeds.

(a) $\lambda_1 = \frac{v}{f_1} = \frac{3 \times 10^8 m/s}{10^{12} s^{-1}} = 3 \times 10^{-4}\ m. = 0.3\ mm.$

$\therefore$ 1.5 mm is five times of the free-space wavelength of violet length.

$\lambda_2 = \frac{v}{f_2} = \frac{3 \times 10^8\ m/s}{88.1 \times 10^6\ s^{-1}} = \frac{300}{88.1} m \doteq 3.41\ m$

so 1.5 mm is much less than radio wave from the campus radio station.

(b) $v = \frac{c}{n} = \frac{3 \times 10^8 m/s}{1.34}$

$\therefore t = \frac{l}{v} = \frac{ln}{c} = \frac{25 \times 10^{-3} m \times 1.34}{3 \times 10^8 m/s} \doteq 1.1 \times 10^{-10}\ s$

3. A light wave.

$$E_z = E_0 \cos(kx - \omega t) = E_0 \cos(\omega t - kx)$$

$$= E_0 \cos\left(10^{15}\pi t - \frac{10^{15}\pi}{0.65c} x\right)$$

$$\therefore \ \omega = 10^{15}\pi \qquad k = \frac{10^{15}\pi}{0.65c}$$

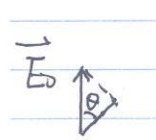
$\therefore$ (a) $f = \dfrac{\omega}{2\pi} = \dfrac{10^{15}\pi}{2\pi} = 5\times10^{14}$ Hz

(b) $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{10^{15}\pi} \times 0.65 \times 3\times10^{8} = 2.6\times10^{-7}$ m.

(c) $n = \dfrac{c}{v} = \dfrac{c}{\lambda f} = \dfrac{3\times10^{8}}{2.6\times10^{-7}\times5\times10^{14}} \doteq 2.3$

(4, 3 pts.**) Malus' Law.**

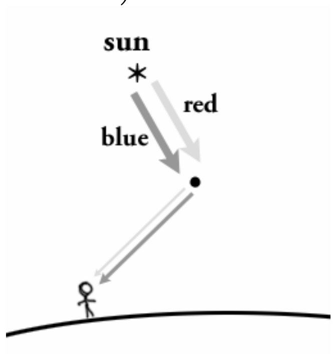
All the light that passes through the polarizer will have its electric field parallel to the polarizer's transmission axis. (This is the "definition" of a linear polarizer.) Any light that will pass through the analyzer will have its electric field parallel to the analyzer's transmission axis. Consider such a wave, whose amplitude after passing through the polarizer is $\vec{E}_0$. Electric fields are vectors, and so we can decompose $\vec{E}_0$ into components parallel and perpendicular to the analyzer axis; these components will have magnitudes $\left|\vec{E}_0\right|\cos\theta$ and $\left|\vec{E}_0\right|\sin\theta$ for the parallel and perpendicular components, respectively (see figure). Only the parallel component goes through the analyzer, and so the resulting intensity is $I \propto \left(\left|\vec{E}_0\right|\cos\theta\right)^2$. Denoting the maximal value of this function as I_0, we can write $I = I_0\cos^2\theta$.



(5, 6 pts.**) The Blue Sky.**

(a) From the problem's statements (i)-(iv) we know that $E_{0s} \propto E_{0i}\dfrac{1}{r}V\lambda^{a}$, where a is some number. Since E_{0s} and E_{0i} have the same dimensions, $\dfrac{1}{r}V\lambda^{a}$ must be dimensionless. The dimensions of V are length-cubed; the dimensions of r are length. Therefore $\left[\lambda^a\right] = \frac{1}{\text{length}^2}$, so $a = -2$. The scattered intensity $I \propto \left(E_{0s}\right)^2$, so $I \propto \lambda^{-4}$.

(b) Consider light emitted by the sun (*****, in the figure) scattered by some scatterer (**•**, in the figure). Some of this light is scattered, radiated in all directions; some of this scattered light reaches the observer looking in the direction of the scatterer (thin lines). How much light is scattered? Some fraction **that depends on the wavelength** in the manner we calculated in part (a). Smaller wavelength light, e.g. blue, is scattered **more** than longer wavelength light, e.g. red. So more blue light intensity is directed towards the observer than green or yellow or red light intensity. The observer sees a blue sky. (What happens to the non-scattered light? It continues in a straight line from the sun, perhaps to some other observer.)



6. Interference.

$\vec{E}_1(x,t) = \vec{E}_0\, e^{j(kx-\omega t-\delta_1)}$, $\vec{E}_2(x,t) = \vec{E}_0\, e^{j(kx-\omega t-\delta_2)}$

$\vec{E} = \vec{E}_1 + \vec{E}_2$

(a) $I = \vec{E}\cdot\vec{E}^*$

$= \vec{E}_0^2 \cdot \left(e^{j(kx-\omega t-\delta_1)} + e^{j(kx-\omega t-\delta_2)}\right)\cdot\left(e^{-j(kx-\omega t-\delta_1)} + e^{-j(kx-\omega t-\delta_2)}\right)$

$= E_0^2\left(1+e^{j(\delta_2-\delta_1)}+e^{j(\delta_1-\delta_2)}+1\right)$

$= E_0^2\left(2+2\cos\Delta\phi\right)$

$= 4I_0\cos^2\frac{\Delta\phi}{2}$

(b). $\lambda_0 = \dfrac{c}{f} = \dfrac{3\times10^{8}\ m/s}{450\times10^{12}s^{-1}} = \dfrac{2}{3}\times10^{-6}m = \dfrac{2}{3}\mu m.$

$\lambda_1 = \dfrac{\lambda_0}{n} = \dfrac{2}{3\times1.46}\ \mu m.$

$10\mu m = 15\lambda_0 = 21.9\lambda_1$

$\therefore \Delta\phi = \dfrac{2\pi}{\lambda_0}(n-1)l$

$= \dfrac{2\pi}{\frac{2}{3}\mu m}\times 0.46\times 10\mu m$

$= 13.8\pi.$

$\therefore I = 4I_0\cos^2\dfrac{\Delta\phi}{2}$

$= 4I_0\cos^2(6.9\pi)$

$= 2I_0\left(1+\cos\dfrac{\pi}{5}\right)$

$\doteq 3.62 I_0$

(7) Anti-reflection coating.

First, just by noting that we "want" destructive interference between the two beams, we would guess that the film thickness should be such that the extra path length $2d$ corresponds to a π phase shift for the wave, or equivalently a $\lambda/2$ distance. Therefore $2d = \lambda/2 = n\lambda_0/2$, where λ_0 is the free space wavelength. Therefore $d = \lambda_0 / 4n_1$. This answer is correct, but we should proceed more carefully: Unlike the simple constructive and destructive interference encountered in class, in which the amplitudes of the two waves were the same, here they are different. So we should "do the math" and see what the interference looks like.

The first wave is $\vec{E}_1 = r\vec{E}_0\exp\left[j(kx-\omega t-\pi)\right]$, where I've included the π phase shift mentioned in the problem. The second wave is $\vec{E}_2 = t\vec{E}_0\exp\left[j(k(x+2d)-\omega t-\pi)\right]$, where I've incorporated the extra path length $2d$ going to and from the substrate, and the π phase shift. They interfere to give:

$\vec{E} = \vec{E}_1+\vec{E}_2 = r\vec{E}_0\exp\left[j(kx-\omega t-\pi)\right]+t\vec{E}_0\exp\left[j(k(x+2d)-\omega t-\pi)\right].$

$\vec{E} = \vec{E}_1+\vec{E}_2 = r\vec{E}_0\exp\left[j(kx-\omega t-\pi)\right](1+)+t\vec{E}_0\exp\left[j(k(x+2d)-\omega t-\pi)\right]$

As usual, $I \propto \vec{E} \bullet \vec{E}^{\,*}$, which gives

$$I \propto \vec{E} = \left|\vec{E}_0\right|^2 \left(r \exp\left[j(kx - \omega t - \pi)\right] + t \exp\left[j(k(x + 2d) - \omega t - \pi)\right]\right)$$

$$\left(r \exp\left[-j(kx - \omega t - \pi)\right] + t \exp\left[-j(k(x + 2d) - \omega t - \pi)\right]\right)$$

$$I \propto r^2 + t^2 + rt \exp\left[-j2kd\right] + rt \exp\left[j2kd\right]$$

$$I \propto r^2 + t^2 + 2rt \cos(2kd)$$

$$I \propto r^2 + t^2 + 2rt \cos\left(2\frac{2\pi n_1}{\lambda_0} d\right)$$

, where we use $k = 2\pi / \lambda$ and the relation between wavelength and λ_0, the free space wavelength

Where is I minimal? The first two terms are fixed, so I is minimal where cosine is minimal, i.e. where its value is -1, which occurs when $\left(2\frac{2\pi n_1}{\lambda_0} d\right) = \pi$. Solving for d:

$$d = \lambda_0 / 4n_1 .$$