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Physics 352; Winter 2008

Problem Set 3: SOLUTIONS

(1) *N*-slit interference.

$$I(\theta) = I_1 \frac{\sin^2(\pi ND\sin\theta/\lambda)}{\sin^2(\pi D\sin\theta/\lambda)},$$

All the maxima, e.g. $\theta = 0$, are equivalent, so we'll consider $\theta = 0$. $I(\theta = 0) \to \frac{0}{0}$, so we apply L'Hopital's rule, differentiating numerator and denominator:

$$I_{\theta\to 0} = I_1 \frac{2[\pi ND\cos\theta/\lambda]\sin(\pi ND\sin\theta/\lambda)\cos(\pi ND\sin\theta/\lambda)}{2[\pi D\cos\theta/\lambda]\sin(\pi D\sin\theta/\lambda)\cos(\pi D\sin\theta/\lambda)} = \frac{0}{0},$$ so we'll apply L'Hopital's rule again. First, it saves some writing to use the identity $\sin(2x) = 2\sin x\cos x$, so the above equation becomes

$$I_{\theta\to 0} = I_1 \frac{[\pi ND\cos\theta/\lambda]\sin(2\pi ND\sin\theta/\lambda)}{[\pi D\cos\theta/\lambda]\sin(2\pi D\sin\theta/\lambda)},$$ from which

$$I_{\theta\to 0} = I_1 \left. \frac{2[\pi ND\cos\theta/\lambda]^2\cos(2\pi ND\sin\theta/\lambda)}{2[\pi D\cos\theta/\lambda]^2\cos(2\pi D\sin\theta/\lambda)} \right|_{\theta=0} = I_1 \frac{[\pi ND/\lambda]^2}{[\pi D/\lambda]^2} = N^2 I_1.$$

Another way to derive this is to Taylor expand $I(\theta)$ for small θ.

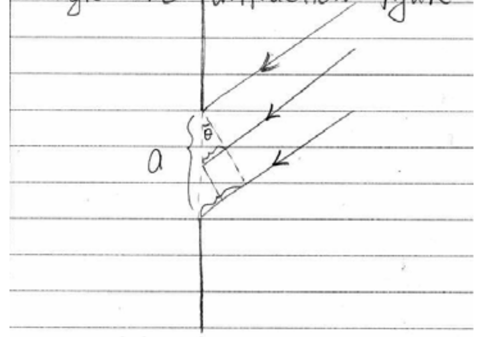
(2, 4 pts.**) Diffraction gratings and extrasolar planets.**

Consider light of wavelength λ. Its first diffraction intensity maximum is located at $\sin\theta = \lambda/D$, and the zero nearest this is at $\sin\theta = \lambda/D + \lambda/ND$, as we know from the *N*-slit intensity function.

We wish to resolve light of wavelength $\lambda + \Delta\lambda$. What does this mean? The "blob" of light present at its first diffraction maximum must be distinguishable from the "blob" of the light of wavelength λ. It's first diffraction maximum should occur at an angle at least as large as the zero of the light of wavelength λ, or else we could not differentiate the two. Therefore, we need $\frac{\lambda + \Delta\lambda}{D} > \frac{\lambda}{D} + \frac{\lambda}{ND}$, or $\Delta\lambda > \frac{\lambda}{N}$. The Doppler effect relates $\Delta\lambda$ and the causative velocity: $\Delta\lambda = \frac{\Delta v}{c}\lambda$. Therefore, the velocity shifts we can observe are: $\frac{\Delta v}{c}\lambda > \frac{\lambda}{N}$, i.e. $\boxed{\Delta v > \frac{c}{N}}$. Note that Δv is independent of λ! For $N = 10^5$, the minimum resolvable Δv is $\Delta v = 3000$ m/s. (As noted in the assignment, this is the minimum resolvable Δv from just looking at the shift of one point on the spectrum; in practice, the entire spectrum can be examined, allowing much more precise determinations of Δv.)

(3, 1 pt.**) Single-slit diffraction figure.**

3. Single-Slit diffraction figure



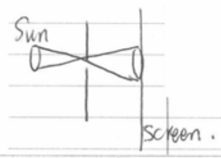
extra path length:

$\Delta l = a\sin\theta$

(4, 4 pts.**) The Sun.**

(a) Hold a ruler at arm's length, measure the apparent diameter d of the moon along the ruler, and measure the length L of your arm. Note that $\tan\theta = d/L$. Since $d \ll L$ you can make use of the small angle approximation $\tan\theta \approx \theta$, which is of course only valid for θ measured in radians. You should find that the angular diameter of the moon (or sun) is about or 0.01 radians, or 0.5 degrees. The same holds true for the sun, whose observation requires a bit more caution.

Of course, there are other ways to measure this as well. Here's one, from our TA (Yan):



(a) -- another approach

use a cardboard with a pinhole on it.
Hold the cardboard perpendicular to the rays of the sun
then put a white paper behind the cardboad and parallel
to it. we will see the image of the sun of the white paper.
measure the size of the image r and the distance between
the cardboard and the white paper l.
we will get $\theta = \frac{r}{l}$ **(Again, this should be 0.01 radians, or 0.5 degrees)**

(b) The angular resolution of our eyes at visible wavelengths ($\lambda \approx 0.5\mu m$) is $\theta \approx \lambda / a \approx \dfrac{0.5\times10^{-6}m}{2\times10^{-3}m} = 10^{-4}$ radians. This is much smaller than the 0.01 radian angular diameter of the sun or moon, and so we **can** resolve these objects.

(c) At radio wavelengths, the resolution of a 2mm aperture (like our pupils) would be $\theta \approx \lambda / a \approx \frac{1m}{2\times10^{-3}m} \approx 10^3$ radians. This is much larger than the angular diameter of the sun or moon, and so we **could not** resolve these objects.

(d) If we want our angular resolution to be < 0.5 degrees, or 0.01 radians, we need $\lambda / a < 0.01$, i.e. $a > \lambda / 0.01$. The frequency of the radio wave $f = 1400\times10^6$ Hz, so the wavelength $\lambda = c / f = 0.21$ m. Therefore, we need $a > 0.21m / 0.01 = 21m$ -- at least a 21 meter separation.

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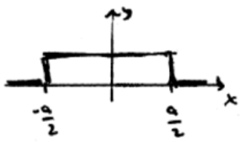
(5) Fourier Transforms and Apertures.

The function: $y(x) = \begin{cases} 1/a, \; if \; |x| \le a/2 \\ 0, \; if \; |x| > a/2 \end{cases}$, as sketched below. Since it's defined piecewise, let's perform the integral piecewise:



$$Y(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(x)\, e^{-jkx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-a/2} 0\, e^{-(\;)} dx + \frac{1}{\sqrt{2\pi}\, a} \int_{-a/2}^{a/2} 1 \cdot e^{-jkx} dx + \int_{a/2}^{\infty} 0 \cdot e^{-(\;)} dx$$

$$= \frac{1}{\sqrt{2\pi}\, a} \int_{-a/2}^{a/2} e^{-jkx}\, dx = \frac{1}{\sqrt{2\pi}} \left(\frac{-1}{jka}\right) \left[e^{-jkx} \right]_{-a/2}^{a/2}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{jka}\right) \left(e^{jk\frac{a}{2}} - e^{-jk\frac{a}{2}}\right) = \frac{1}{\sqrt{2\pi}} \frac{2}{ka} \sin\left(\frac{ka}{2}\right)$$

{noting that

$$\sin(x) = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right)$$

Thus $\underline{|Y(k)|^2 = \frac{2}{\pi} \left[\frac{\sin(ka/2)}{ka} \right]^2}$, which has the same "$\left(\frac{\sin\beta}{\beta}\right)^2$" form as the diffraction pattern intensity of a single 1D slit.

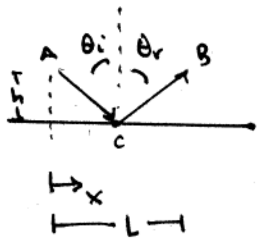
If we wish, we can define a relation between this purely mathematical k and the physical geometry of diffraction — let's say $k = 2\pi \sin\theta / \lambda$, so $|Y(k)|^2 = \frac{2}{\pi} \left(\frac{\sin(\pi a \sin\theta/\lambda)}{2\pi a \sin\theta/\lambda} \right)^2$,

or $|Y(\theta)|^2 = \frac{1}{2\pi} \frac{\sin^2\beta}{\beta^2}$, with $\beta = \frac{\pi a}{\lambda} \sin\theta$ as usual.

This is exactly the same angular dependence as the intensity of diffraction from a 1D slit.

(6) Law of Reflection.

pt. c @ horizontal position x with respect to A.



medium of index of refr. n.

time from A to c $\quad t_1 = \dfrac{\sqrt{h^2+x^2}}{v} \quad (v = c/n)$

" " c to B $\quad t_2 = \dfrac{\sqrt{h^2+(L-x)^2}}{v}$

Total time $t = t_1 + t_2$ is extremal $\Rightarrow \dfrac{dt}{dx} = 0$

$$\frac{dt}{dx} = \frac{1}{2v}\frac{2x}{\sqrt{h^2+x^2}} + \frac{-2(L-x)}{2v\sqrt{h^2+(L-x)^2}} = 0$$

$$\rightarrow \underbrace{\frac{x}{\sqrt{h^2+x^2}}}_{\sin\theta_i} = \underbrace{\frac{L-x}{\sqrt{h^2+(L-x)^2}}}_{\sin\theta_r}$$

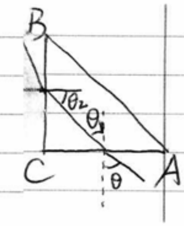
$$\rightarrow \boxed{\theta_i = \theta_r} \quad ✓$$

(7) A prism.

7\. A prism.

assume the incident angle is θ

then as shown in the figure left.



$$n\sin\theta_1 = \sin\theta$$

$$\theta_1 + \theta_2 = \frac{\pi}{2}$$

$$\therefore \sin\theta_2 = \cos\theta_1 = \sqrt{1-\sin^2\theta_1} = \sqrt{1-\frac{\sin^2\theta}{n^2}}$$

in order to have light transmitted through face BC

$$n\sin\theta_2 < 1$$ **(i.e. no total internal reflection)**

$$\therefore n\sqrt{1-\frac{\sin^2\theta}{n^2}} < 1$$

$$n^2 - \sin^2\theta < 1$$

$$n^2 < 1+\sin^2\theta$$ **n must be less than** $\sqrt{1+\sin^2\theta}$ **for transmission to occur**

$$\theta < 90^\circ$$

$$\therefore n < \sqrt{2}$$

so if Mr. K do not want to pay you

he shall choose $n \geq \sqrt{2}$.

$\therefore$ the smallest n is $\sqrt{2} \doteq 1.414$

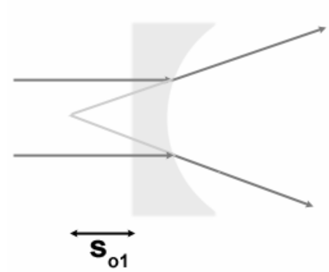
(8) Galilean Telescope.

(a) Lens 1: $R_1 = \infty$, $R_2 = +200$ cm (concave, right side)

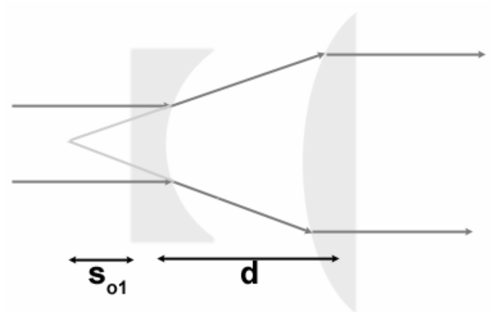
Lens 2: $R_1 = +450$ cm (convex, left side), $R_2 = \infty$.

(b) Let's consider each lens separately.

First consider lens 1. The focal length f_1 is given by $\frac{1}{f_1} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$. Therefore: $f_1 = \left[(n-1)\left(0 - \frac{1}{200\text{cm}}\right)\right]^{-1} = -400\text{cm}$, using $n = 1.5$. The "object" length $s_{o1} = \infty$, since the rays are parallel to the optical axis. Applying the thin lens equation: $\frac{1}{s_{o1}} + \frac{1}{s_{i1}} = \frac{1}{f_1}$ yields $s_{i1} = f_1 = -400\text{cm}$. There is a virtual image to the left of the lens – see figure.

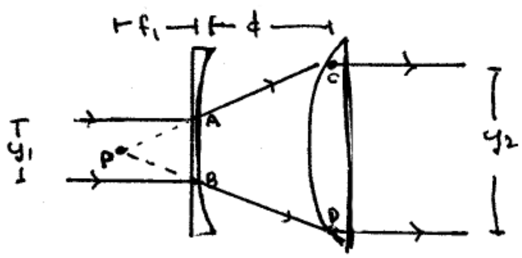


Now consider lens 2, with its "object" being the image of lens 1, located $|s_{i1}| = |f_1|$ to the left of lens 1. We want the image of lens 2 to be rays parallel to the optical axis. So for lens 2, $s_{i2} = \infty$. The object length for lens 2 is $s_{o2} = d - s_{i1}$, where d is the separation between the lenses (see Figure).



The focal length $f_2 = \left[(n-1)\left(\frac{1}{450\text{cm}} - 0\right)\right]^{-1} = 900\text{cm}$. Thin lens: $\frac{1}{s_{o2}} + \frac{1}{s_{i2}} = \frac{1}{f_2}$, so $\frac{1}{d - s_{i1}} + 0 = \frac{1}{f_2}$, and $d = s_{i1} + f_2 = -400\text{cm} + 900\text{cm}$. Therefore $d = 500\text{cm}$. This is the desired separation. Note that $d = f_1 + f_2 = f_2 - |f_1|$.

(c)



(d) From the diagram, ΔPAB and ΔPCD are similar triangles.

Thus $\frac{y_1}{|f_1|} = \frac{y_2}{d+|f_1|} \rightarrow \frac{y_2}{y_1} = \frac{d+|f_1|}{|f_1|} = \frac{f_2}{|f_1|}$

Therefore $\frac{y_2}{y_1} = \frac{900}{400} = \frac{9}{4}$.