Prof. Raghuveer Parthasarathy
University of Oregon; Winter 2008

Physics 352: Problem Set 6

Due date: Wednesday, Feb. 20, **5pm**.
Reading: K&K Chapters 1-2.

Comments:

- The first few problems are mathematical exercises related to tools and techniques we'll use in the course. I recommend doing them soon. The last few problems deal with issues of probability and entropy.
- For a few problems, you may use the following **definite integral of a Gaussian function**:

$$\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$$

(1, 5 pts**). The Gamma Function.** The gamma function, $\Gamma(x)$, is defined by:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$. We'll make use of it later in the course.

(a, 1 pt.**)** Show that $\Gamma(1) = 1$

(b, 2 pts.**)** Show that $x\,\Gamma(x) = \Gamma(x+1)$, and therefore that $\Gamma(n+1) = n!$ for integer n (the exclamation mark indicates n-factorial). Thus the gamma function provides an "extension" of factorials from the integers to all the real numbers. (*Hint*: Integrate by parts.)

(c, 2 pts.**)** Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. (*A big hint*: make a substitution $u = \sqrt{t}$, and then use Gaussian integral given at the top of the assignment.) You <u>may not</u> use any expressions from a table of integrals.

(2, 4 pts.**) Averages.** For a "discrete" distribution in which each possible state has probability p_i, the **average** (or **mean**) value of some function A is $\langle A \rangle = \sum_i A_i p_i$, where A_i is the value of A in state i. (Note that all the p_i must add to 1; in other words $\sum_i p_i = 1$.) For example, if I flip a coin, there are two states, heads and tails, each with probability $p_1 = 1/2$. Suppose I take 2 steps to the left if I flip "tails" and 3 to the right if I flip "heads," so that $A_{heads} = +3$ and $A_{tails} = -2$. Then $\langle A \rangle$, my average motion per step is $\langle A \rangle = \sum_i A_i p_i = (-2)\frac{1}{2} + (3)\frac{1}{2} = +\frac{1}{2}$.

(a, 1 pt.**)** Consider a normal six-sided die, in which the probability of any face coming up is $1/6$. Suppose you gain \$$(n-3)$ for each roll of the die, where n is the number of dots on the die face you roll. On average, how much money will you make per roll?

We often deal with **continuous** probability distributions $p(x)$ – the number $p(x)dx$ gives the probability that x will have a value in the range $[x, x+dx]$. The **average** (or **mean**) value of a function $f(x)$ is given by

$\langle f(x) \rangle = \int f(x) p(x) dx$, where the integral is over the entire domain of *x*.

Consider a Gaussian probability distribution:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

. (This has the property that $\int_{-\infty}^{\infty} p(x)dx = 1$, as it should, which you don't have to verify.)

(b, 1 pt.**)** Show that $\langle x \rangle = 0$. (*Hint*: you can evaluate this without doing any integrals – *draw* the integral!)

(c, 2 pts.**)** Show that $\langle x^2 \rangle = \sigma^2$. (σ is the *standard deviation* of the distribution.) You may use the Gaussian integral provided at the top of the assignment, but must derive all other expressions. (*Hint*: integration by parts, with $u = x$, may be useful.)

(3, 1 pt.**)** **More probabilities.** The combinatoric argument that gives us the binomial distribution is easily extended to objects with more than two possible states – for example, a six-sided die. Consider an object with t possible states – for the die, $t = 6$. If we roll our die N times, what is the number of configurations with n_1 ones, n_2 twos, ... and n_6 sixes? It is

$$\boxed{\Omega = \frac{N!}{n_1!n_2!...n_t!}}$$

. (You don't have to prove this.)

So, for example, if we roll a die 8 times and want to know how many ways there are to get 1 one, 5 twos, 0 threes, 0 fours, 2 fives, and 0 sixes, it's $\frac{8!}{1!5!0!0!2!0!} = \frac{8\times7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1\times2} = 168$ possible configurations. (12222255, 21222255, 22122255, 51225222, etc.).

Consider a four-sided die, rolled five times. How many configurations are there with 2 ones, 0 twos, 3 threes, and 0 fours?

Yes, this is a boring problem. I'm assigning it because it leads us to problem 5, which is interesting. First another "simple" observation: Given the above distribution, the probability of rolling face "i" is $\boxed{p_i = \frac{n_i}{N}}$ -- for example, for our 8 rolls of a 6-sided die above, $p_1 = \frac{1}{8}$, $p_2 = \frac{5}{8}$, $p_3 = 0$, etc. Note that $\sum p_i = 1$ -- the probabilities must add to one. (Similarly, note that $\sum n_i = N$.)

(4, 5 pts.**) Probability and entropy.** In class, we've defined entropy: $\sigma = \ln \Omega$, where Ω is the number of available states. Here, we'll derive a different, but equivalent, expression for the entropy that relates it to probabilities. (This turns out to be surprisingly useful, as we'll see later.) Consider as in Problem 3 N elements each with t possible states, so that $\Omega = \frac{N!}{n_1!n_2!...n_t!}$. **Show that** for the usual large N, the entropy $\boxed{\sigma = -N\sum_i p_i \ln(p_i)}$, where the sum runs over all t states.

Here's a sketch of what to do:

- Start from $\sigma = \ln \Omega$.
- Simplify the logarithm using product and quotient rules. Make use of Stirling's approximation: $\ln(N!) \approx N \ln N - N$, and note that $\sum n_i = N$.
- Rewrite your answer in terms of probabilities, p_i, using the relation from Problem 4. You should find something like $\sigma = N \ln N - \sum_i p_i N \ln\left(p_i N\right)$. Simplify your relation. Note that $\sum p_i = 1$, and also note that constants can be factored out of sums – i.e. $\sum_i a x_i = a \sum_i x_i$.

(5, 4 pts.**) The binary spin system.** Consider, as discussed in class, a binary spin system with $N-r$ "up" spins each with energy $+\epsilon$ and r "down spins" each with energy $-\epsilon$. We'll again define the spin excess $s = N - 2r$, so that $r = \frac{N-s}{2}$ and $N - r = \frac{N+s}{2}$. Consider $|s| \ll N$, and show that the entropy $\sigma(E) = N \ln 2 - \frac{E^2}{2\epsilon^2 N}$, where E is the total energy. Note that we've done this derivation in class, except for some algebraic details which you can now fill in.

(6, 7 pts.**) Distribution of the multiplicities for two systems.** In class we considered two initially isolated systems, with energies E_1 and E_2, brought into "thermal contact" so that E_1 and E_2 could vary as long as the sum $E = E_1 + E_2$ remained fixed. We argued that in general, the degeneracy of the supersystem would be maximal for some E_1 between 0 and E – this followed from assuming that Ω_1 and Ω_2 are increasing functions of E_1 and E_2, respectively. This is a very reasonable assumption – there should be more states available as we increase the energy of a system – and it's consistent with what our "example" systems of a particle-in-a-box and a hydrogen atom show. We've discussed a very specific example – the binary spin system. Here, let's consider a "somewhat" specific example.

Consider two systems for which the number of available states varies as a power of the energy. In other words: $\Omega_1 = aE_1^{\alpha}$ and $\Omega_2 = bE_2^{\beta}$, where $\alpha > 1$ and $\beta > 1$ and a and b are constants. The systems are brought into thermal contact. Again, the total energy E is fixed.

(a, 1 pt.**)** From the fundamental definition of temperature, determine expressions for τ_1 and τ_2 as functions of E_1 and E_2, respectively.

(b, 2 pts.**)** Find the equilibrium (i.e. most probable) energies of system 1 and system 2 by determining the overall degeneracy, Ω, and finding where its derivative is zero. Your answer will depend on α, β, and E.

(c, 1 pt.**)** What happens to your answer for part (b) if $\frac{\beta}{\alpha}$ gets large? Does this make sense?

(d, 2 pts.**)** Show that the equilibrium energies you found in part (b) satisfy the relation $\tau_1 = \tau_2$, using your relations from (a).

(e, 2 pts.**)** Suppose $a = b$ and $\alpha = \beta$. Suppose initially, System 1 has energy $3E/4$ and System 2 has energy $E/4$. The two systems are brought into thermal contact. If we think only about the contents of System 1: Has the entropy of System 1 increased or decreased relative to its isolated state? (You should find that it has decreased.) Mr. K. has heard people whispering on the street that entropy always increases, and so is bothered by your claim. What do you say to him?

You might find the following plot, of Ω vs. E / E_1 for $a = b$ and $\alpha = \beta$, interesting. You don't have to do anything with it.