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Physics 352; Winter 2008

Problem Set 6: SOLUTIONS

1. The Gamma Function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

(a) $\Gamma(1) = \int_0^\infty t^{1-1} e^{-t} dt$

$$= \int_0^\infty e^{-t} dt = -e^{-t}\Big|_0^\infty = -(0-1) = 1$$

(b) $x\Gamma(x) = x\int_0^\infty t^{x-1} e^{-t} dt$

$$= \int_0^\infty x t^{x-1} e^{-t} dt$$

$$= \int_0^\infty e^{-t} dt^x$$

$$= e^{-t} t^x \Big|_0^\infty + \int_0^\infty t^x e^{-t} dt$$

$$= \int_0^\infty t^x e^{-t} dt$$

$$= \Gamma(x+1)$$

$\therefore \Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1)$

$$\cdots$$

$$= n(n-1)\cdots 1\cdot\Gamma(1)$$

$$= n!$$

(c) $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt$

set $u = \sqrt{t}$ then $t = u^2$ $dt = 2u\,du$.

$$\therefore \Gamma(\tfrac{1}{2}) = \int_0^\infty \frac{1}{u} e^{-u^2} \cdot 2u\,du$$

$$= 2\int_0^\infty e^{-u^2} du$$

$$= \int_{-\infty}^\infty e^{-u^2} du$$

$$= \sqrt{\pi}.$$

2. Averages

(a) $\langle A \rangle = \sum_{i=1}^{6} A_i P_i$

$= \sum_{n=1}^{6} (n-3) \times \frac{1}{6} = 0.5 \$$

(b) $\langle x \rangle = \int x P(x)\, dx$

$= \int_{-\infty}^{\infty} dx \frac{x}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$

the integrand is an odd function, so the integral is zero

that is $\langle x \rangle = 0$

(c) $\langle x^2 \rangle = \int_{-\infty}^{\infty} dx \frac{x^2}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$

$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2\sigma^2} dx$

$= \frac{-\sigma^2}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, d e^{-x^2/2\sigma^2}$

$= \frac{-\sigma}{\sqrt{2\pi}} x e^{-x^2/2\sigma^2} \Big|_{-\infty}^{\infty} + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} dx$

$= \frac{\sigma}{\sqrt{2\pi}} \cdot \sqrt{2}\sigma \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}\sigma}\right)^2} d\left(\frac{x}{\sqrt{2}\sigma}\right)$

$= \frac{\sigma^2}{\sqrt{\pi}} \cdot \sqrt{\pi}$

$= \sigma^2$

3. More Probabilities

$$\Omega = \frac{N!}{\prod_{i=1}^{t} n_i!} = \frac{5!}{2! \times 0! \times 3! \times 0!} = 10$$

4 Probability and entropy

$$\begin{aligned}
\sigma &= \ln \Omega \\
&= \ln \frac{N!}{n_1! n_2! \cdots n_t!} \\
&= \ln N! - \sum_{i=1}^{t} \ln (n_i!) \\
&\doteq N \ln N - N - \sum_{i=1}^{t} \ln (N p_i)! \\
&= N \ln N - N - \sum_{i=1}^{t} \left(N p_i \ln (N p_i) - N p_i \right)
\end{aligned}$$

$\cdot p_i = 1)$

$$\begin{aligned}
&= N \ln N - \sum_{i=1}^{t} N p_i \ln (N p_i) - N + N \\
&= N \ln N - N \sum_{i=1}^{t} p_i \left(\ln N + \ln p_i \right) \\
&= -N \sum_{i=1}^{t} p_i \ln p_i
\end{aligned}$$

5. The binary spin system

$$E = (N - r)\varepsilon + r(-\varepsilon)$$
$$= (N - 2r)\varepsilon = s\varepsilon$$
$$\therefore S = \frac{E}{\varepsilon}$$

$$\sigma(E) = \ln \Omega$$
$$= \ln \frac{N!}{(N-r)!\, r!}$$
$$= \ln N! - \ln \left(\frac{N-S}{2}\right)! - \ln \left(\frac{N+S}{2}\right)!$$
$$\doteqdot N \ln N - N - \frac{N-S}{2} \ln \frac{N-S}{2} + \frac{N-S}{2} - \frac{N+S}{2} \ln \frac{N+S}{2} + \frac{N+S}{2}$$
$$= N \ln N - \frac{N-S}{2}\left(\ln \frac{N}{2} + \ln\left(1 - \frac{S}{N}\right)\right) - \frac{N+S}{2}\left(\ln \frac{N}{2} + \ln\left(1 + \frac{S}{N}\right)\right)$$
$$= N \ln 2 - \frac{N-S}{2}\left(-\frac{S}{N} - \frac{1}{2}\left(-\frac{S}{N}\right)^2\right) - \frac{N+S}{2}\left(\frac{S}{N} - \frac{1}{2}\left(\frac{S}{N}\right)^2\right)$$
$$= N \ln 2 - \frac{S^2}{2N}$$
$$= N \ln 2 - \frac{E^2}{2\varepsilon^2 N}$$

6 Distribution of multiplicities for two systems.

$\Omega_1 = aE_1^{\alpha}$ $\quad \Omega_2 = bE_2^{\beta}$ $\qquad \alpha > 1, \ \beta > 1.$

(a) $\frac{1}{T} = \frac{d\sigma}{dE}$

$\sigma_1 = \ln\Omega_1 = \ln a + \alpha \ln E_1$

$\sigma_2 = \ln\Omega_2 = \ln a + \beta \ln E_2$

$\therefore \frac{1}{\tau_1} = \frac{d\sigma_1}{dE_1} = \frac{\alpha}{E_1}$ $\qquad \frac{1}{\tau_2} = \frac{d\sigma_2}{dE_2} = \frac{\beta}{E_2}$

$\therefore \tau_1 = \frac{E_1}{\alpha}$ $\qquad \tau_2 = \frac{E_2}{\beta}$

(b) $\Omega = \Omega_1\Omega_2 = ab\, E_1^{\alpha} E_2^{\beta} = ab\, E_1^{\alpha}(E-E_1)^{\beta}$

$$\frac{\partial\Omega}{\partial E_1} = ab\left(\alpha E_1^{\alpha-1}(E-E_1)^{\beta} - \beta E_1^{\alpha}(E-E_1)^{\beta-1}\right) = 0$$

$$\therefore \alpha E_1^{\alpha-1}(E-E_1)^{\beta} = \beta E_1^{\alpha}(E-E_1)^{\beta-1}$$

$\therefore E_1 = \frac{\alpha E}{\alpha+\beta}$ $\qquad E_2 = \frac{\beta E}{\alpha+\beta}$

(c) if $\frac{\beta}{\alpha}$ gets large then $E_2 \gg E_1$.

it makes sense, for Ω_1 & Ω_2 are increasing function of E_1 & E_2, so $E_2 \gg E_1$ to make Ω get its maximum.

(d) $\tau_1 = \frac{E_1}{\alpha}$ $\tau_2 = \frac{E_2}{\beta}$

$\therefore \tau_1 = \frac{E}{\alpha+\beta} = \tau_2$

(e) $a=b$, $\alpha=\beta$

at initial state

$$E_1 = \frac{3}{4}E \ , \ E_2 = \frac{E}{4}$$

at equilibrium

$$E_1 = E_2 = \frac{1}{2}E$$

$\sigma_1 = \ln a + \alpha \ln E_1$

$\therefore$ the entropy of system 1 decreased

However, the entropy of the whole system increased.

we can see,

initially,

$$\begin{aligned}\sigma &= \sigma_1 + \sigma_2 \\ &= \ln a + \alpha \ln E_1 + \ln a + \alpha \ln E_2 \\ &= 2\ln a + \alpha \ln (E_1 E_2) = 2\ln a + \alpha \ln \left(\frac{3}{16}E^2\right)\end{aligned}$$

equilbrium,

$$\begin{aligned}\sigma' &= \sigma_1' + \sigma_2' \\ &= \ln a + \alpha \ln \left(\frac{E}{2}\right) + \ln a + \alpha \ln \left(\frac{E}{2}\right) \\ &= 2\ln a + \alpha \ln \left(\frac{1}{4}E^2\right) > \sigma.\end{aligned}$$

that's what people whispered, the entropy of the whole system always increases, while entropy of one part of the system can decrease.