Prof. Raghuveer Parthasarathy
University of Oregon; Winter 2008

Physics 352: Problem Set 7

Due date: Wednesday, Feb. 27, **5pm**.
Reading: K&K Chapter 3.

(1, 2 pts.**)   An electrostatic potential.**   Your cells maintain electrostatic potential differences between their inside and outside. Consider a potential difference of $V = +100 \, \text{mV}$ such that the inside is at a higher potential than the outside, and a singly-charged ion like Na+ (charge q = $+1.6 \times 10^{-19}$ C). At standard body temperature, what is the probability that the ion will be inside the cell? (Boltzmann's constant $k_B = 1.38 \times 10^{-23}$ Joules/K.) [Note: this is a ridiculously simplified view of cellular ion potentials – we'll get better later...]

(2, 4 pts.**)   The CO_2 molecule, revisited.**   Last quarter, we examined the classical vibrational motion of a CO_2 molecule and found two normal mode frequencies, f_1 and f_2. (Don't worry if you were not in my 351 course.) This result is true quantum mechanically as well. Corresponding to each normal mode are many possible states, each with a different energy separated by energy hf, where f is the normal mode frequency and h is Planck's constant, 6.63×10^{-34} Joules-seconds. Specifically, the allowed energy levels for mode 1 are

$$E_{n1} = \left(n_1 - \frac{1}{2} \right) hf_1, \text{ where } n_1 \text{ is a positive integer } (n_1 = 1,2,3,...),$$

and for mode 2 are

$$E_{n2} = \left(n_2 - \frac{1}{2} \right) hf_2, \text{ where } n_2 \text{ is another positive integer } (n_2 = 1,2,3,...).$$

The lowest energy state of any mode ($n = 1$) is referred to as the "ground state." The normal mode frequencies are measured to be $f_1 = 40$ THz and $f_2 = 70.5$ (1 THz = 10^{12} Hz).

Suppose one examines a population of CO_2 molecules in an interstellar cloud. One can measure the population of molecules in the first excited state ($n = 2$) relative to the ground state ($n = 1$) for each normal mode. (One does this by examining the absorption of light at different wavelengths). Call this population ratio r. ~~Suppose we find that r for mode 1 is 0.1, and r for mode 2 is 0.0001.~~ **REVISED VERSION:** Suppose we find that the ratio of the r values for the two modes is $r_{mode1} / r_{mode2} = 5.8$. From this information, **determine the temperature** of the cloud, in Kelvin. (Boltzmann's constant $k_B = 1.38 \times 10^{-23}$ Joules/K.)

(3, 4 pts.) The Partition Function. Consider a system with Q possible states.

(a, 2 pts.) Show that in the high temperature limit ($\tau \to \infty$), each state is equally probable; show also that the partition function $Z \to Q$.

(b, 2 pts.) Show that in the zero temperature limit ($\tau \to 0$), the probability for finding the system in any state but the ground (lowest energy) state goes to zero; show also that the partition function $Z \to 1$.

(4, 7 pts.) A four-bead polymer chain. A *polymer* molecule is a chain of repeating molecular units. (Rubber is one of many common examples.) Typically the chain is very long, thousands or millions of "monomer" units, but we'll consider a four-bead polymer chain[1] on a 2D grid. (See the figure, which shows two possible microstates of the system.) Each bead has to sit at a grid site, and two beads can't occupy the same site. The chain can bend at $90°$ angles at each site. Imagine that the monomers at each of the two endpoints of our chain are special – they repel one another in such a way as to give the polymer an energy $E = \dfrac{\varepsilon}{d^2}$, where d is the distance between endpoints in grid units. (For the two configurations shown, $d = 1$ and $d = \sqrt{5}$.) Note that the highest energy state has $d = 1$.
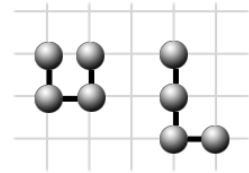
(a, 2 pts.) Draw all the microstates of the system – i.e. all possible configurations – and calculate their energy. (Don't count rotations of the entire polymer as distinct states – think about why.)

(b, 1 pt.) Tabulate the degeneracy of the "macrostate" with end-to-end distance d, for all distinct d values.

(c, 3 pts.) Calculate the mean end-to-end separation, $\langle d \rangle$ as a function of temperature.

(d 1 pt.) Plot $\langle d \rangle$ as a function of τ/ε.

(5, 6 pts.) Paramagnetism and the Curie Law. Consider, as in class, N spins each of which can point either up or down along some axis, e.g. an axis defined by an applied magnetic field $\boldsymbol{B}$. Each spin has a "magnetic moment" m and therefore an energy due to the magnetic field of $E_1 = \pm mB$ (positive if the spin is parallel to the field and negative if antiparallel). The total energy of the system is therefore $E = -mBs$, where s is the *spin excess* defined in class, i.e. the number of up spins minus the number of down spins.

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[1] States of four- and six-bead chains are also discussed in the Dill and Bromberg text, on reserve.

The **fractional magnetization, M**, of our system of spins is defined by: $M = \langle s \rangle / N$, where $\langle s \rangle$ is the mean (expectation) value of s. Note that if $M = 1$, $\langle s \rangle = N$, i.e. all the spins point in the same direction.

Using the expression derived in class (or the text[2]) for the entropy of a spin system for $|s| \ll N$, **find the equilibrium value of M as a function of m, B, and the temperature τ**. You should find that $M \propto 1/\tau$ – i.e. the magnetization drops inversely with temperature – known as the **Curie Law** of paramagnetism. *Hint*: Show that for $|s| \ll N$, $\sigma(E) = \sigma_0 - E^2 / 2m^2 B^2 N$.

(6, 7 pts.**) Calibrating an optical trap.** We often wish to **apply and measure forces**. As we saw innumerable times last term, "any" force (F) that restores an object to equilibrium is roughly spring-like: force and displacement from equilibrium (x) are related by $F = -kx$, from which it follows via the general relation $F = -\dfrac{dU}{dx}$ that $U = \dfrac{1}{2}kx^2$. The symbol k denotes the spring constant of the spring. Macroscopically, we can apply and measure forces with springs or other "familiar" tools. How can we deal with forces at **microscopic** scales – for example, manipulating a cloud of atoms, or measuring the force exerted by a protein motor?

As I've mentioned in class, **optical traps** provide a powerful tool for microscopic manipulation. A laser focused to a point attracts objects to the center of the focus – the light itself is an intangible "spring." Before we can use this spring to measure forces, we need to know its spring constant. How do we measure k? Statistical mechanics!

Particle displacement -- 3μm silica bead
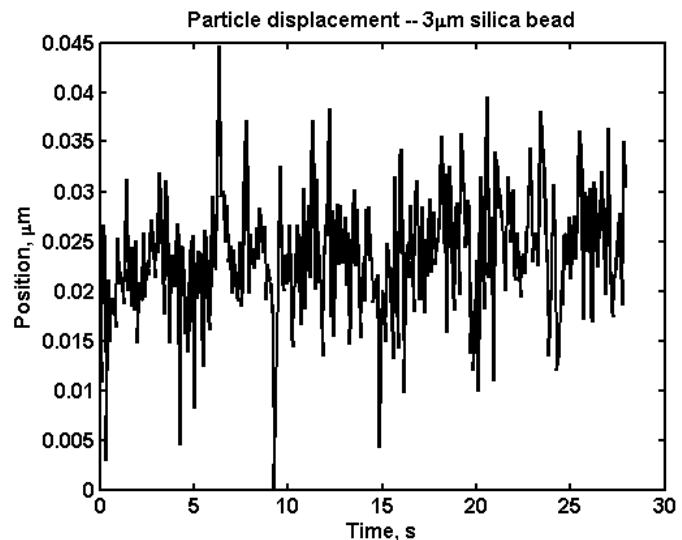
In this problem I'll present **real experimental data,** from the Parthasarathy Lab, from which **your task is to determine k**. The setup consists of a microsphere (a 3-μm diameter glass bead) trapped in an optical trap. Taking movies and performing various image analysis algorithms, we determine the location of the particle center at each "frame" of our movie. The x-coordinate of these positions looks like:

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[2] The textbook defines the spin excess as 2s, rather than s. This problem, by the way, is the same as K&K #2.2.

I'll do **one** step of the analysis for you: From the above "x" values, we can determine the probability distribution $p(x)$ by sorting the data into "bins" of some convenient size. p for each bin is given by the number of data points in each bin, normalized so that $\sum p_i = 1$. (This is very easy to do, but I'm doing it for you just so you don't have to write a computer program to sort the data yourself.) Here's the result:

x, μm	$p(x)$
0.00000	0.00893
0.00448	0.00595
0.00896	0.03274
0.01344	0.10417
0.01792	0.27679
0.02239	0.30952
0.02687	0.16964
0.03135	0.06548
0.03583	0.02381
0.04031	0.00298

(a, 1 pt.) The position data is measured with respect to some arbitrary point – i.e. $x = 0$ **is not** the equilibrium position. What is? (i.e. what is $\langle x \rangle$?)

(b, 6 pts.) Using the data in the above table, and the Boltzmann distribution, **determine the spring constant of the trap. Express your k in units of Newtons/meter.** I am intentionally not telling "spelling this out" for you – you know all the necessary physics, and your task is to figure out how to implement it!

Some hints and comments:

- Boltzmann's constant $k_B = 1.38 \times 10^{-23}$ Joules/K. The temperature during the experiment was about 23°C (296 K). One micron (μm) is 10^{-6} m.
- Graph things! For example, you may wish to graph $p(x)$ vs. x.
- What would you expect $p(x)$ for a spring-like energy function to look like?
- Hopefully, you realize from the above question that $p(x)$ is not a linear function of anything useful. When analyzing data, it is often useful to change scales, especially so that linear relations emerge. What would you expect $\ln(p(x))$ vs. x to look like? Is this any "better?"
- Be careful with units.

[Comment added in email: *Graphing -- and deducing what to graph -- are the keys to this problem, which is a very "real world" exercise. If you think about how to graph the data, the answer can emerge as a simple property of the graph. For example, if I plot the distance I bicycle as a function of the time I spend on the bike, my (average) speed emerges as the slope of that line. If I plot the log of the distance vs. the square root of the time, nothing insightful emerges.*]