

Physics 352: Problem Set 8

Due date: Wednesday, Mar. 5, 5pm.

Reading: K&K Chapter 3.

(1, 3 pts.) Thermal energy scales. We've noted in class that the fundamental temperature, τ , has the same dimensions as energy, E , and that for a system with an unbounded spectrum of energies, like a quantum oscillator, the mean energy $\approx \tau$ in the high-temperature limit. We'll soon discuss the equipartition theorem, which again will relate mean energy to τ . One often uses energy and temperature "interchangeably," referring to the temperature that corresponds to a particular energy, via the relations $\tau \approx E$ and $\tau = k_B T$, where T is the conventional temperature and Boltzmann's constant $k_B = 1.38 \times 10^{-23}$ Joules/Kelvin. Here are some short questions on "characteristic" temperatures.

- (a, 1 pt.)** A characteristic energy scale for removing electrons from atoms is ~ 1 eV; 1 eV is the energy of 1 electron (charge 1.6×10^{-19} Coulombs) in a 1 Volt potential. The universe used to be very hot, and has been cooling while expanding since the Big Bang. Free protons and electrons scatter light very effectively compared to bound atoms; the universe became relatively transparent to thermally generated photons when stable atoms could exist, which occurred about 13.7 billion years ago, 400,000 years after the big bang. What was the temperature of the universe at this point? (Light from this period is what one sees in the cosmic microwave background, by the way; it's called the "surface of last scattering.")
- (b, 1 pt.)** Magnetism is due to coupling between electron spins – the spins energetically prefer to be aligned, but thermal energy prefers them to be randomized. At temperatures above the "Curie Temperature," T_C , thermal energy is greater than the coupling; below, it's less. (We'll be more rigorous about this later in the course.) T_C for Nickel is 630 K. Estimate the coupling between the spins in Nickel, in eV.
- (c, 1 pt.)** Recall the "particle in a box" whose energy levels we discussed in class. Consider an electron (mass $m = 9.1 \times 10^{-31}$ kg) in a cube with sides of length L . Roughly how small would L have to be for the energy difference between the lowest and second-lowest energy levels to be greater than typical "room temperature." Express your answer in nanometers (1 nm = 10^{-9} m) – your answer illuminates the essential reason that nanostructures are interesting; in nanostructures, quantum mechanical properties like discrete energy levels are inescapable.

(2, 4 pts.) **Particle wavelengths.** Quantum mechanically, all objects (particles) have a wavelength given by $\lambda = h/p$, where h is Planck's constant and p is the particle's momentum.

(a, 2 pts.) For a particle of mass m with kinetic energy equal to the temperature τ (see #1), derive an expression for the wavelength in terms of m , τ , and h .

(b, 1/2 pt.) Calculate λ , in meters, for an electron (mass 9.1×10^{-31} kg) at room temperature.

(c, 1/2 pt.) Calculate λ , in meters, for Mr. K. (mass 14 kg) at room temperature.

(d, 2 pts.) Liquid helium has a density of about 125 kg/m^3 (about 1/8 the density of water). Helium atoms have a mass of 4 amu ($4 \times 1.7 \times 10^{-27}$ kg). Imagine that the density of liquid helium is constant with respect to temperature (not actually true). At what temperatures (i.e. above or below what critical value) would the wavelength of helium be larger than the mean distance between atoms? This is a very crude estimate of the superfluid transition temperature, corresponding to helium's transition to a "quantum liquid" with zero viscosity at 2.2K. This argument fails for hydrogen, by the way, for reasons we'll see later.

(3, 6 pts.) **A 1-particle gas.** Consider one particle of mass m in a 3D box of sides L , and hence volume $V=L^3$ at temperature τ . The energy levels of the particle are given by

$$E(n_1, n_2, n_3) = \frac{\hbar^2 \pi^2}{2mV^{2/3}} (n_1^2 + n_2^2 + n_3^2), \text{ where each } n \text{ is a positive integer.}$$

(a, 4 pts.) Calculate the partition function, Z . Convert your sum to a product of sums; convert each sum to an integral to evaluate it (valid for the regime $\frac{\hbar^2 \pi^2}{\tau 2mV^{2/3}} \ll 1$, which, from problem 2, corresponds to a box that is large compared to the quantum wavelength). You may find the following integral useful: $\int_0^\infty e^{-y^2} dy = \sqrt{\pi}/2$. You should find that Z is proportional to V .

(b, 1 pt.) Define a quantity n_Q via $Z = n_Q V$, using your answer to part a. Show that your expression for n_Q has dimensions of *concentration*, i.e. length^{-3} . (A concentration is a number per unit volume.)

(c, 1 pt.) n_Q is called the **quantum concentration**. Why? (Look back at your answer to 2a.)

A suggestion: If you're having trouble with Problem 3, first prove the following relation:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} [f(m)g(n)h(p)] = \left(\sum_{m=1}^{\infty} f(m) \right) \left(\sum_{n=1}^{\infty} g(n) \right) \left(\sum_{p=1}^{\infty} h(p) \right),$$

where m , n , and p are integers and f , g , and h are some arbitrary functions. f is only a function of m , not n or p , etc. You may find it helpful to write out several terms of the triple sum: $f(1)g(1)h(1) + f(2)g(1)h(1) + f(3)g(1)h(1) + \dots f(1)g(56)h(1) + f(1)g(57)h(1) + \dots$

(4, 5 pts.) **The equipartition theorem.** Often the energy of a system, E , varies quadratically as a function of some parameter. For our 1D particle-in-a-box, $E \propto n^2$, where n is the quantum number; for a mass-on-a-spring (from last quarter) $E \propto x^2$, where x is the displacement from equilibrium. Suppose some dimensionless variable s characterizes some generic “degree of freedom,” like position along some axis. You can think of s as characterizing a particular state with that s value. Suppose that $E(s) = cs^2$, where c is some constant.

(a, 1 pt.) Write down an expression for the mean energy of the system, $\langle E \rangle$ at temperature τ . Leave your answer in terms of sums over all possible s values.

(b, 4 pts.) Consider the “classical” case of high temperature, or small spacing between levels, in which we can approximate s as being able to take a continuum of values from $-\infty$ to ∞ , turning our sums into integrals. Show that $\langle E \rangle = \frac{1}{2}k_B T$. This is the “equipartition theorem.”

It states, roughly speaking, that each classical degree of freedom “gets” $\frac{1}{2}k_B T$ of energy. Note that this result is independent of the energy scale, c . You may find some of the definite integrals evaluated in earlier problem sets useful.

(5, 4 pts.) **A relativistic gas.** We showed in class that for a non-relativistic gas, for which $E \propto V^{-2/3}$ as in #3, pressure, energy, and volume are related by $P = \frac{2}{3} \frac{U}{V}$. For relativistic particles (e.g. photons, or anything moving near the speed of light), $E \propto V^{-1/3}$. Show that for a relativistic gas, $P = \frac{1}{3} \frac{U}{V}$.

(6, 8 pts.) **Zipper Problem – Kittel & Kroemer #3.7.** In this problem, you’ll examine the behavior of an N -link “zipper” in which it costs energy to open a link. *Hint:* Remember geometric series.