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Physics 352; Winter 2008

Problem Set 8: SOLUTIONS

(1, 3 pts.**) Thermal energy scales.**

(a) Atoms are stable when, roughly, $k_B T < 1eV$. 1 eV = $1.6\times10^{-19} J$, so $T = \left(1.6\times10^{-19} / 1.38\times10^{-23}\right)K$ = **T = 11000 Kelvin.**

(b) At T = T_C, thermal energy approx. equals the coupling: $k_B T_C =$ Coupling, so Coupling, in eV = $k_B T_C / e = 1.38\times10^{-23}\times 630 / 1.6\times10^{-19}$ eV = 0.054 eV.

(c)

c). $E(n_x, n_y, n_z) = \frac{\hbar^2 \pi^2}{2m\, L^2}(n_x^2+n_y^2+n_z^2)$.

$n_x, n_y, n_z = 1, 2, 3, \cdots$

∴ the lowest energy is

$E_1 = E(1,1,1) = \frac{\hbar^2\pi^2}{2m\, L^2}\times 3$

the second-lowest energy is

$E_2 = E(1,1,2) = E(1,2,1) = E(2,1,1) = \frac{\hbar^2\pi^2}{2mL^2}\times 6$

∴ $\Delta E = E_2 - E_1 = \frac{3\hbar^2\pi^2}{2mL^2} \doteq \tau = k_B T \qquad T \doteq 21°C = 294\ K$

$$\therefore L = \left(\frac{3\hbar^2\pi^2}{2mk_BT}\right)^{1/2} = \left(\frac{3\times(6.63\times10^{-34}\ Js)^2}{8\times9.1\times10^{-31}\times1.38\times10^{-23}\times294\ kg\cdot J}\right)^{1/2}.$$

$$\doteq 6.68\times10^{-9} m$$

$$= 6.68\ nm.$$

2. Particle wavelengths

a)
$$E = \frac{p^2}{2m} = \frac{1}{2m}\frac{h^2}{\lambda^2} \approx \tau$$

$$\therefore \lambda \approx \frac{h}{\sqrt{2m\tau}}$$

b) for an electron $(m = 9.1\times10^{-31}\,kg)$ at room temperature $T = 294\,k$

$$\lambda \approx \frac{6.63\times10^{-34}\,Js}{\sqrt{2\times9.1\times10^{-31}\,kg \times 1.38\times10^{-23}\,J/k \times 294\,k}}$$

$$\doteq 7.7\times10^{-9}\,m = 7.7\,nm.$$

c) for $m = 14\,kg$, $T = 294\,K$

$$\lambda = \frac{6.63\times10^{-34}\,Js}{\sqrt{2\times14\,kg \times (1.38\times10^{-23}\times294)\,J}}$$

$$\doteq 1.94\times10^{-24}\,m.$$

(d) $\rho = 125\,kg/m^3$, $m_{He} = 4\times1.7\times10^{-27}\,kg$.

$\therefore$ Particle number density

$$n = \frac{\rho}{m_{He}} = \frac{125\,kg/m^3}{4\times1.7\times10^{-27}\,kg} = 1.84\times10^{28}\,m^{-3}$$

$\therefore$ the mean distance between atoms

$$l \approx \left(\frac{1}{n}\right)^{1/3} = \left(\frac{1}{1.84\times10^{28}}\right)^{1/3} \doteq 3.8\times10^{-10}\,m.$$

$$\lambda = \frac{h}{\sqrt{2m\tau}} \approx l$$

$$\therefore T = \frac{h^2}{2ml^2 k_B} = \frac{(6.63\times10^{-34})^2}{2\times4\times1.7\times10^{-27}\times(3.8\times10^{-10})^2\times1.38\times10^{-23}}$$

$$\doteq 16\,K$$

(3, 6 pts.**) A 1-particle gas.**

$$E(n_1, n_2, n_3) = \frac{\hbar^2 \pi^2}{2 m V^{2/3}} \left(n_1^2 + n_2^2 + n_3^2 \right) \qquad n_1, n_2, n_3 = 1, 2, 3 \cdots$$

a) $$Z = \sum_i e^{-\varepsilon_i/\tau}$$

$$= \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} e^{-\frac{\hbar^2 \pi^2 (n_1^2+n_2^2+n_3^2)}{2mV^{2/3} k_B T}}$$

$$= \left(\sum_{n_1=1}^{\infty} e^{-\frac{\hbar^2\pi^2}{2mV^{2/3}\tau} n_1^2} \right) \left(\sum_{n_2=1}^{\infty} e^{-\frac{\hbar^2\pi^2}{2mV^{2/3}\tau} n_2^2} \right) \left(\sum_{n_3=1}^{\infty} e^{-\frac{\hbar^2\pi^2}{2mV^{2/3}\tau} n_3^2} \right)$$

because the box is large compared to quantum wavelength,

$$\therefore \quad L \gg \lambda = \frac{h}{\sqrt{2m\tau}}$$

$$\therefore \quad \frac{\hbar^2 \pi^2}{2mV^{2/3}\tau} \ll 1$$ so we can convert the sum into integral

$$\therefore \sum_{n=1}^{\infty} e^{-\frac{\hbar^2\pi^2}{2mV^{2/3}\tau} n_1^2} \simeq \int_0^{\infty} d\left[\left(\frac{\hbar^2\pi^2}{2mV^{2/3}\tau} \right)^{-\frac{1}{2}} x \right] e^{-x^2}$$

$$= \left(\frac{\hbar \pi}{\sqrt{2m\tau}\, V^{1/3}} \right)^{-1} \int_0^{\infty} e^{-x^2} dx$$

$$= \frac{\sqrt{2\pi m \tau}\, V^{1/3}}{2\hbar\pi} = \frac{\sqrt{2\pi m\tau}}{h} V^{1/3}$$

similarly,

$$\sum_{n_2=1}^{\infty} e^{-\frac{\hbar^2\pi^2}{2mV^{2/3}\tau} n_2^2} = \sum_{n_3=1}^{\infty} e^{-\frac{\hbar^2\pi^2}{2mV^{2/3}\tau}} = \frac{\sqrt{2\pi m\tau}\, V^{1/3}}{\hbar \pi}$$

$$\therefore Z = \frac{(2\pi m \tau)^{3/2}}{8\hbar^3\pi^3} V$$, it's proportional to V.

b) $$Z = n_Q V$$

$\because$ from (a), we get $$n_Q = \frac{(2\pi m\tau)^{3/2}}{8\hbar^3\pi^3} = \frac{(2\pi m \tau)^{3/2}}{h^3}$$

so it has dimensions

$$\frac{(M \cdot M \cdot L^2 \cdot T^{-2})^{3/2}}{(M \cdot L^2 \cdot T^{-2})^3 \cdot T^3} = \frac{1}{L^3}$$

that is n_Q has dimensions of concentration.

(c) Note from your answer to Problem 2 that $n_Q \approx 1/\lambda^3$, where λ is the particle wavelength. Therefore when the concentration $\approx n_Q$, the mean spacing between particles is similar to their quantum-mechanical wavelength.

4. The equipartition theorem.

$$E(s) = cs^2$$

(a) $$\langle E \rangle = \sum_s \frac{1}{z} E(s) e^{-E(s)/\tau}$$

$$= \frac{\sum_s (cs^2 e^{-cs^2/\tau})}{\sum_s e^{-cs^2/\tau}}$$

(b) consider the "classical" case of high temperature

$$\langle E \rangle \approx \frac{\sqrt{\frac{\tau}{c}} \int_{-\infty}^{\infty} \tau x^2 e^{-x^2} dx}{\sqrt{\frac{\tau}{c}} \int_{-\infty}^{\infty} ds\, e^{-s^2}}$$

$$= \frac{\tau \frac{1}{2}\sqrt{\pi}}{\sqrt{\pi}}$$

$$= \frac{1}{2}\tau = \frac{1}{2} k_B T$$

(5, 4 pts.**)** **A relativistic gas.**

A relativistic gas. generally, we've shown that $P = -\left.\frac{\partial U}{\partial V}\right|_\sigma$,

so $P = -\sum_i p_i \left.\frac{\partial E_i}{\partial V}\right|_\sigma$, from $U = \sum_i p_i E_i$ (mean energy).

$E_i = aV^{-1/3}$, where a is some constant.

so $$\frac{\partial E_i}{\partial V} = -\frac{1}{3} a V^{-4/3} = -\frac{1}{3}\frac{aV^{-1/3}}{V} = -\frac{1}{3}\frac{E_i}{V}$$

$$\rightarrow \quad P = -\sum_i p_i \left(-\frac{1}{3}\right)\frac{E_i}{V} = +\frac{1}{3}\frac{1}{V}\sum_i p_i E_i$$

$$= \boxed{P = \frac{1}{3}\frac{U}{V}}$$

Note that this is a nearly identical derivation to our in-class non-relativistic gas.

(6) Zipper Problem

A general comment: This problem illustrates how the partition function and Boltzmann relation provide information about physical systems. Note, either from your own solutions or those below, that there's no need to determine an explicit function for the entropy, or its relation to energy! That's the great thing about the partition function and the Boltzmann distribution -- they "contain" this information and provide an alternate, and generally simpler, way of extracting information about the system. All we need to do to be able to exploit them is to be able to enumerate the states of the system and the states' energies – drawing them can be a good place to start!

A good way to start this problem is to think of a few microstates of the zipper and <u>draw them</u>. You'll find that you can easily enumerate all the states – there's a state with zero links open, with energy 0; a state with 1 link open with energy 1ε; a state with 2 links open with energy 2ε; a state with 3 links open with energy 3ε; up to a state with N links open with energy Nε. There is no need to try to tabulate numbers of accessible states and get at the entropy – you're not asked to, and more importantly this information is "contained" in the partition function, which is much easier (in general) to calculate. So calculate the partition function:

(a) $Z = \sum_i e^{-E_i/\tau}$, where the sum is over all states. We know how to enumerate the states – the number of open links, s, is a great label:

$$Z = \sum_{s=0}^{N} e^{-s\varepsilon/\tau}$$

. Note that the energy $E(s) = s\varepsilon$. Defining $\beta \equiv 1/\tau$,

$$Z = \sum_{s=0}^{N} e^{-\beta s\varepsilon} = \sum_{s=0}^{N} \left(e^{-\beta\varepsilon}\right)^{s}$$

, a geometric series with the sum $\boxed{Z = \dfrac{1-e^{-\beta\varepsilon(N+1)}}{1-e^{-\beta\varepsilon}}}$.

(b) We want to calculate $\langle s \rangle$, the average number of open states.

Fundamentally, $\langle s \rangle = \sum_i sp_i = \frac{1}{Z}\sum_s s\exp(-\beta\varepsilon s)$.

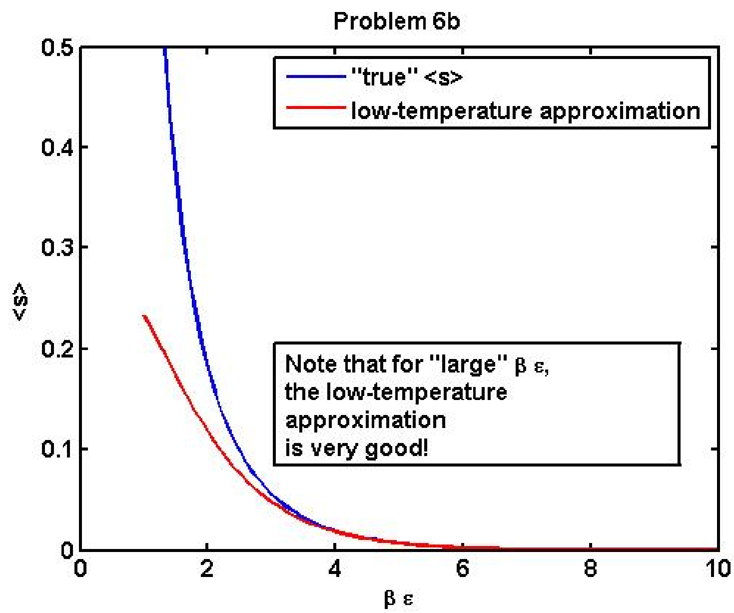
We are interested in $\langle s \rangle$ at low temperature. Many people took the $\tau \to 0$ limit, for which one can show quite easily that $\langle s \rangle = 0$. I won't take off points for this, but this isn't what I wanted. There's a difference between **low** temperature and **zero** temperature.

What I was really going for was:

$$< s >= \frac{1}{Z}\sum_{s=0}^{N} se^{-s\varepsilon/\tau} = \left(\frac{1-e^{-\beta\varepsilon}}{1-e^{-\beta(N-1)\varepsilon}}\right)\sum_{s=0}^{N} se^{-\beta s\varepsilon}$$

At low temperature, βε>>1; let's delete all the terms that contain higher powers of $e^{-\beta\varepsilon}$ than the first power. The denominator of the prefactor just becomes 1; the first term of the sum is zero anyway, so we only keep the s = 1 term. This gives us:

$\boxed{< s >\approx \left(1-e^{-\beta\varepsilon}\right)e^{-\beta\varepsilon}}$. As we can see by plotting it, this low temperature approximation is pretty good.



Problem 6b
"true" <s>
low-temperature approximation
Note that for "large" β ε,
the low-temperature
approximation
is very good!
<s>
β ε
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10