- SECOND-ORDER DISPERSION

Go back to III-3 and keep 2^{nd}-order term

$$(k(\omega)-k_\ell) \approx (\omega-\omega_\ell)k_\ell' + \tfrac{1}{2}(\omega-\omega_\ell)^2 k_\ell'' \equiv \nu k_\ell' + \tfrac{1}{2}\nu^2 k_\ell''$$

$\nu \equiv \omega - \omega_\ell$

Then

$$\mathcal{E}_\ell(z,t) = \int_{B_\ell} \frac{d\omega}{2\pi} \mathcal{E}(z,\omega)\, e^{i(k(\omega)-k_\ell)z}\, e^{-i(\omega-\omega_\ell)t}$$

$$\partial_z \mathcal{E}_\ell(z,t) = \partial_z \int_{B_\ell} \frac{d\nu}{2\pi} \mathcal{E}(z,\omega_\ell+\nu)\, e^{i(\nu k_\ell' + \frac{1}{2}\nu^2 k_\ell'')z}\, e^{-i\nu t}$$

$$= \int \frac{d\nu}{2\pi} \mathcal{E}(z,\omega_\ell+\nu)\left[i\nu k_\ell' + \tfrac{i}{2}\nu^2 k_\ell''\right] e^{i(\nu k_\ell' + \frac{1}{2}\nu^2 k_\ell'')z}\, e^{-i\nu t}$$

7B.76 ✓ $$= i k_\ell'\, i \frac{\partial}{\partial t}\mathcal{E}_\ell(z,t) + \tfrac{1}{2}k_\ell''(-i)\frac{\partial^2}{\partial t^2}\mathcal{E}_\ell(z,t)$$

$$\boxed{\left(\partial_z + k_\ell'\partial_t + \tfrac{i}{2}k_\ell''\partial_t^2\right)\mathcal{E}_\ell(z,t) = 0}$$

- Moving Frame : Define $\tau = t - k_\ell' z = t - z/v_{g\ell}$

Then

$$\boxed{i\frac{\partial}{\partial z}\mathcal{E}_\ell = \frac{k_\ell''}{2}\frac{\partial^2}{\partial \tau^2}\mathcal{E}_\ell}$$

Looks like the Schrödinger Equation with z & τ swapped.

HW Leads to Group Velocity Dispersion (GVD) ⇒ Packet Spreading

HW (III-23, 2015) — Typical Values for Silica Optical Fiber

For $\lambda = 532$ nm $k_\ell'' \approx 60\ ps^2/km$

$\lambda = 1270$ nm ≈ 0

$\lambda = 1550$ nm $\approx -15\ ps^2/km$

(Agrawal pg. 385 NL Fiber Optics)

• Nonlinear Schrödinger Equation (Classical E&M)

Add an instantaneous nonlinear term to the RHS.
(That is, we treat the time response of the linear terms correctly, but not the NL terms.)
(Don't follow Grynberg Comp. 7B.1 who uses a 2-level medium)

Go back to the wave equation and consider a single field with broad spectrum. $\mathcal{E}_e \equiv \mathcal{E}_1$

III-15;16 $$\frac{\partial}{\partial z}\mathcal{E}_1 = e^{-ik_1 z}\,\frac{i\omega_1}{2\epsilon_0 nc}\,\mathcal{P}^{\omega_1}_{NL}(z) \qquad \sim e^{-i\omega_1 t}$$

slowly varying

$$\tilde{\mathcal{E}}_1 = \mathcal{E}_1(t)\,e^{ik_1 z}$$

not in text:

$$P^{(3)} = \epsilon_0 \chi^{(3)} E^3 \quad \{\text{not monochromatic}\}$$
$$= \epsilon_0\chi^{(3)}\left[\tilde{\mathcal{E}}_1 e^{-i\omega_1 t} + \tilde{\mathcal{E}}_1^* e^{+i\omega_1 t}\right]^3$$
$$= \epsilon_0\chi^{(3)}\left[\tilde{\mathcal{E}}_1^2 e^{-i2\omega_1 t} + \tilde{\mathcal{E}}_1^{*2} e^{i2\omega_1 t} + 2|\tilde{\mathcal{E}}_1|^2\right]\left[\tilde{\mathcal{E}}_1 e^{-i\omega_1 t} + \tilde{\mathcal{E}}_1^* e^{i\omega_1 t}\right]$$
$$= \epsilon_0\chi^{(3)}\Big[\underbrace{\tilde{\mathcal{E}}_1^2\tilde{\mathcal{E}}_1^*}_{\tilde{\mathcal{E}}_1|\tilde{\mathcal{E}}_1|^2} + 2|\tilde{\mathcal{E}}_1|^2\tilde{\mathcal{E}}_1\Big]e^{-i\omega_1 t} + \underbrace{\text{other terms}}_{e^{-i3\omega_1 t}}$$

HW 2 $$= \epsilon_0\chi^{(3)}\,3\,|\mathcal{E}_1|^2\mathcal{E}_1\,e^{-i\omega_1 t}\,e^{ik_1 z}$$

(even though $\mathcal{E}_1(t)$ is time dependent, all other terms are out-of-band.)

$$\Rightarrow \boxed{\partial_z \mathcal{E}_1 = \frac{i\omega_1}{2nc}\,\chi^{(3)}\,3\,|\mathcal{E}_1|^2\mathcal{E}_1}$$

(Grynberg has 2 instead 3. He seems wrong.)

set $\mathcal{E}_1 \equiv \mathcal{E}$

- Now add back in the Linear Dispersion

$$\frac{\partial}{\partial z}\mathcal{E} + k_1'\frac{\partial}{\partial t}\mathcal{E} + \frac{ik_1''}{2}\frac{\partial^2}{\partial t^2}\mathcal{E} = \frac{i\omega_1}{2nc}\,3\chi^{(3)}|\mathcal{E}|^2\mathcal{E}$$

$\mathcal{E} \equiv \mathcal{E}_1$ $\qquad k_1' = \frac{1}{v_g}$

end L4 '15

B • Define $B \equiv k_1''/2$ (GVD)

Γ Define $\Gamma \equiv \frac{\omega_1}{cn}\frac{3}{2}\chi^{(3)}$ (Nonlinearity)

Define $\tau \equiv t - k_0' z = t - z/v_g$

$$\Rightarrow \boxed{i\frac{\partial}{\partial z}\mathcal{E}(z,\tau) - B\frac{\partial^2}{\partial \tau^2}\mathcal{E}(z,\tau) = -\Gamma\,|\mathcal{E}(z,\tau)|^2\,\mathcal{E}(z,\tau)} \quad \text{NLSE}$$

looks like $i\frac{\partial}{\partial t}\psi = -B\frac{\partial^2}{\partial x^2}\psi + V\psi$, where $V = -\Gamma|\psi|^2$ and $B < 0$

• <u>Self-phase Modulation</u> (SPM)

For a pulse that is not too short in a medium that is not too long (so there is no linear spreading)

$$\frac{\partial}{\partial z}\mathcal{E} \simeq i\Gamma|\mathcal{E}|^2\mathcal{E}$$ still depends on τ (not steady state)

– Note $|\mathcal{E}|^2$ is constant : proof: $\frac{\partial}{\partial z}|\mathcal{E}|^2 = \mathcal{E}^*\frac{\partial}{\partial z}\mathcal{E} + c.c. = 0$

Therefore $|\mathcal{E}(z,\tau)|^2 \equiv I(\tau)$

Therefore $\mathcal{E}(z,\tau)$ evolves only in its <u>phase</u>.

$$\star \left[\ \mathcal{E}(z,\tau) = \mathcal{E}(0,\tau)\exp\left[i\Gamma I(\tau) z\right]\right.$$

$$\mathcal{E}(L,\tau) = \mathcal{E}(0,\tau)e^{i\Phi(\tau)}$$

where $\Phi(\tau) = \Gamma \cdot I(\tau)\, z$ (given z)

<u>Example</u>: $I(\tau) = \text{constant} = I_0$, Then $\Phi = \Gamma I_0 z$

Putting back into the full field:

$$E = \mathcal{E}e^{ik_0 z}e^{-i\omega t}$$
$$= \mathcal{E}(0)e^{i\Gamma I_0 z}e^{ik_0 z}e^{-i\omega t}$$
$$= \mathcal{E}(0)e^{i(k_0+\Gamma I_0)z}e^{-i\omega t}$$

$$k_0 + \Gamma I_0 = \frac{\omega_1}{c}n + \frac{\omega_1}{c}\frac{3\chi^{(3)}}{2n}I_0 = \frac{\omega_1}{c}\left(n + \frac{3\chi^{(3)}}{2n}I_0\right)$$

- We see that $\frac{3}{2}\frac{\chi^{(3)}}{n} I_0$ gives a correction to the Refractive Index. It is common to denote

$$\boxed{n_2 \equiv \frac{3\chi^{(3)}}{2n}}$$ (does not have unit of refr. index)

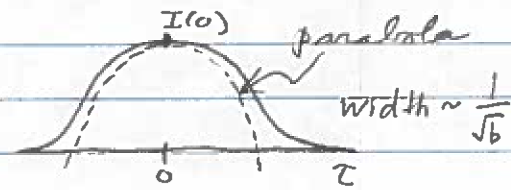
Grynberg 7B.2.1 p 582

Then $\boxed{n \to n + n_2 I_0}$

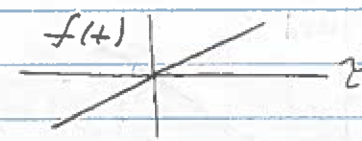
- Time-Dependent SPM ⇒ Chirp $e^{-i(\omega t - \Phi(t))}$

Define Instaneous Frequency $\boxed{f = -\dot{\Phi}(\tau) = -\Gamma z \frac{\partial}{\partial \tau} I(\tau)}$

Expand $I(\tau) = I(0)\left[1 - b\tau^2 + \ldots\right]$ for small τ from center.

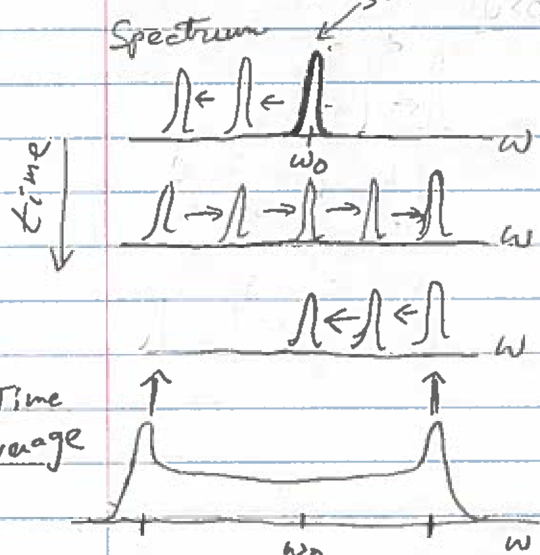


$f(t) = -\dot{\Phi}(\tau) = I_0 2b\Gamma z\tau$
linear chirp near center.

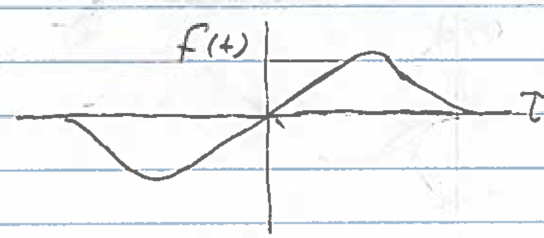


Generally:

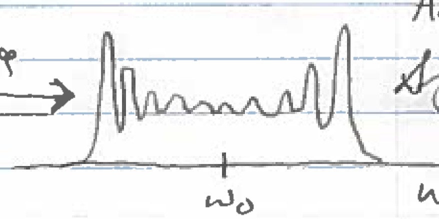
H/W



⇐



Interference Effect ⟶



Actual Spectrum

↑ ↑

Dwells longest at turn-around points, (like a H.O.)

★• For Anomalous Dispersion, NL Chirp Compensates Spreading

p21 Linear Dispersion : $f_L(t) \cong \frac{2Bz}{\sigma^4}\tau$ (long pulse)

NL Chirp : $f_{NL}(t) \cong I_0 2b\Gamma z\tau$ pulse width $\approx \frac{1}{\sqrt{b}} \sim \sigma$

$B<0$, Sum : $f_{NL} - f_L = 2\left(I_0 b\Gamma - \frac{|B|}{\sigma^4}\right) z\tau$ $\frac{1}{b} \approx \sigma^2$

$= 0 \Rightarrow \frac{I_0\Gamma}{\sigma^2} = \frac{|B|}{\sigma^4}$ or $I_0\Gamma = \frac{|B|}{\sigma^2}$

Estimate of pulse width $\sim \sigma = \sqrt{\frac{|B|}{I_0\Gamma}}$

For $B = \frac{1}{2}k_0'' < 0$

For a pulse of duration $\boxed{\sigma \approx \sqrt{\frac{|k_0''|}{2I_0\Gamma}}}$ we expect

to see a balancing between spreading and compressing, so a stable pulse.

• Soliton Solution

MW2 ☆ $i\frac{\partial}{\partial z}\mathcal{E} - B\frac{\partial^2}{\partial\tau^2}\mathcal{E} = -\Gamma|\mathcal{E}|^2\mathcal{E}$ has a solution

$\mathcal{E}(z,\tau) = E_0 e^{i\delta k' z}\,\mathrm{sech}\left(\frac{\tau}{T_s}\right)$ } plug into PDE

check: $i\frac{\partial}{\partial z}\mathcal{E} = -\delta k'\mathcal{E}$

Mma ⇒ $(B + \delta k' T_s^2) = 0$ AND $(2B + \Gamma E_0^2 T_s^2) = 0$ to satisfy

same as above! if $B<0$ $\boxed{T_s = \sqrt{\frac{-B}{\delta k'}}}$ added $\boxed{\delta k' = \frac{-B}{T_s^2}}$ $\boxed{E_0^2 = +\frac{2|B|}{\Gamma T_s^2}}$ $\boxed{B<0}$

Peak intensity must have a certain value.

Scaled "Energy" $= \int_{-\infty}^{\infty} |\mathcal{E}|^2 d\tau = 2T_s E_0^2 = 4|B|/\Gamma$, fixed by medium.

- Properties of the Soliton

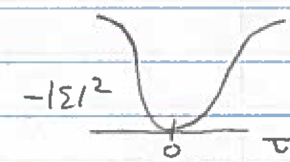
1. Propagates with Group Velocity $v_g = \left.\frac{d\omega}{dk}\right|_{\omega_0}$, which is the same as the v_g in the linear medium. (Contrast this to soliton in two-level medium on resonance where the v_g can be $\frac{1}{1000}v_g$ in linear case.)

2. There is a Nonlinear Phase that continues to accumulate due to nonlinear refraction

3. The soliton is stable because linear dispersion $\frac{k_0''}{2}\frac{\partial^2 \mathcal{E}}{\partial \tau^2}$ is balanced by NL Refraction

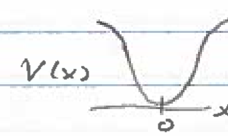
4. Can rewrite NLSE as (for $B<0$)

$$i\partial_z \mathcal{E} = -|B|\partial_\tau^2 \mathcal{E} - \Gamma|\mathcal{E}|^2\mathcal{E}$$



same as

$$i\partial_t \psi = -\frac{\hbar^2}{2m}\partial_x^2 \psi + V\psi$$



So $-\Gamma|\mathcal{E}|^2$ acts like a confining Potential
So there can be bound states. (solitons)