

Supplementary Information

Melting temperature changes during slip across subglacial cavities drive basal mass exchange

by Rempel, Meyer, and Riverman

This Supplementary Information includes brief further discussion of i) equilibrium phase behavior, ii) the vertical force balance, iii) variations in basal equilibrium temperature, iv) perturbed conductive temperature profiles, and v) the potential for englacial phase changes to modify thermal profiles.

Equilibrium Phase Behavior – Equations (1) and (2)

Equation (1) of the main text describes how the equilibrium temperature along an ice–liquid interface is affected by liquid pressure P and the normal component of the ice stress tensor at the interface σ_n . The somewhat involved arguments leading to this result for the general case of a non-hydrostatic ice stress state can be found in the referenced citations (e.g. Kamb, 1961; Paterson, 1973; Sekerka and Cahn, 2004) and will not be repeated here. However, it is both instructive and relatively straightforward to consider the idealized case of phase equilibrium conditions along an ice–liquid interface for which the homogeneous ice pressure is P_i . At equilibrium, the chemical potential μ (i.e. Gibbs free energy per molecule) of each phase must be the same. Considering small deviations of μ_i in the ice and μ_l in the liquid water from a reference state at temperature T_0 and pressure P_0 , the Gibbs–Duhem equation allows the change in chemical potential in each phase to be expressed using the leading-order terms in a Taylor series describing departures from those reference temperature and pressure conditions so that

$$\left. \frac{\partial \mu_l}{\partial T} \right|_{P_0} (T_{\text{eq}} - T_0) + \left. \frac{\partial \mu_l}{\partial P} \right|_{T_0} (P - P_0) \approx \left. \frac{\partial \mu_i}{\partial T} \right|_{P_0} (T_{\text{eq}} - T_0) + \left. \frac{\partial \mu_i}{\partial P_i} \right|_{T_0} (P_i - P_0) .$$

Here, the left side is the change in chemical potential in the liquid phase obtained by altering the temperature from T_0 to T_{eq} and the liquid pressure from P_0 to P , while the right side is the change in chemical potential in the solid ice obtained by altering the temperature from T_0 to T_{eq} and the ice pressure from P_0 to P_i . The change in chemical potential with temperature at constant pressure is the specific entropy s , and the change

27 in chemical potential with pressure at constant temperature is the specific volume, which
 28 is the inverse of density ρ . Making these substitutions and rearranging leads to

$$(s_l - s_i)(T_{\text{eq}} - T_0) \approx \frac{1}{\rho_i}(P_i - P_0) - \frac{1}{\rho_l}(P - P_0) ,$$

29 where the subscripts l and i refer to properties of liquid water and ice, respectively. The
 30 difference in specific entropy between liquid water and ice can be expressed as the ratio of
 31 the latent heat of fusion \mathcal{L} to the reference temperature T_0 , yielding (after further algebraic
 32 rearrangements)

$$T_{\text{eq}} - T_0 \approx \frac{T_0(\rho_i - \rho_l)}{\rho_l \rho_i \mathcal{L}} \left[P - P_0 + \frac{\rho_l}{\rho_l - \rho_i}(P_i - P) \right] . \quad (\text{S1})$$

33 Following substitution of equation (2) from the main text for the Clapeyron slope C_0
 34 and the ice normal stress σ_n for P_i , equation (S1) generalizes to equation (1) in the main
 35 text when the ice stress state at the interface cannot be approximated as hydrostatic.
 36 In non-hydrostatically stressed solids, the Gibbs free energy and its associated chemical
 37 potential are not well defined (e.g. Kamb, 1961). The careful analysis that leads to the
 38 result provided in equation (1) of the main text proceeds from the recognition that the
 39 dominant influence of the ice stress tensor on phase equilibrium conditions arises from
 40 the virtual work performed against σ_n as a result of the change in specific volume that
 41 accompanies the change in phase (Paterson, 1973, p. 366). Although the other principal
 42 components of the ice stress tensor do not have a significant influence on the equilibrium
 43 melting temperature, their departure from σ_n can have important dynamic consequences.
 44 Common glaciological examples of such non-hydrostatic stress states occur at the surface of
 45 a collapsing borehole (e.g. Nye, 1953) or an R-channel containing liquid at a pressure that is
 46 lower than the ice pressure (i.e. defined as one third the trace of the stress tensor) so that a
 47 deviatoric radial stress drives creep closure at a rate that is compensated in steady state by
 48 turbulent melting (Röthlisberger, 1972). Importantly, in such a system the force balance
 49 constraint requires that changes in liquid pressure be balanced by changes in normal stress
 50 and equation (1) correctly describes how this drives changes in the equilibrium melting
 51 temperature that follow the Clapeyron slope. Where temperature measurements coincide
 52 with liquid pressure measurements in boreholes that access the glacier bed, as recognized
 53 by Nye (1953), typically the normal stress in the ice is equal to the liquid pressure and
 54 transient pressure changes are expected to produce temperature changes that follow the
 55 Clapeyron slope (e.g. Andrews and others, 2014; Huss and others, 2007).

56 Vertical Force Balance – Equations (3) and (4)

57 Equation (3) of the main text contains a simple analysis of the vertical force balance over
 58 a representative portion of the glacier base, with horizontally projected area \mathcal{A} . Two

59 key assumptions deserve mention. First, the area is chosen to be of sufficient size that
 60 the effects of vertical shear stresses on its boundaries (e.g. those that are associated with
 61 vertical components of ice flow) are negligible in comparison with the ice weight that must
 62 be supported over \mathcal{A} , namely

$$\int_{\mathcal{A}} \rho_i g H \, d\mathcal{A} . \quad (\text{S2})$$

63 Second, the entire basal surface is separated by liquid from underlying bed materials (re-
 64 ferred to in the main text as minerals, noting that they may be comprised of both bedrock
 65 and unconsolidated sediments), and so it cannot support a shear stress. This implies that
 66 the vertical component of the suitably integrated stress in the ice normal to the basal
 67 surface can be equated with the glacier weight. With the basal ice surface at elevation
 68 $z = b(x, y)$ for some function b and horizontal Cartesian coordinates x and y , the unit
 69 outward normal to the basal ice surface can be expressed as

$$\hat{\mathbf{n}} = \frac{\left(\frac{\partial b}{\partial x} \hat{\mathbf{x}}, \frac{\partial b}{\partial y} \hat{\mathbf{y}}, -\hat{\mathbf{z}} \right)}{\sqrt{\left(\frac{\partial b}{\partial x} \right)^2 + \left(\frac{\partial b}{\partial y} \right)^2 + 1}} ,$$

70 where the Cartesian unit vectors are $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. This implies that the magnitude of the
 71 vertical component (i.e. $\hat{\mathbf{z}}$) of normal stress on a basal area element is

$$\frac{\sigma_n}{\sqrt{\left(\frac{\partial b}{\partial x} \right)^2 + \left(\frac{\partial b}{\partial y} \right)^2 + 1}} ,$$

72 and recognizing that the ratio between the size of a basal area element and its projection
 73 on the horizontal plane is

$$\frac{da}{d\mathcal{A}} = \sqrt{\left(\frac{\partial b}{\partial x} \right)^2 + \left(\frac{\partial b}{\partial y} \right)^2 + 1} ,$$

74 the magnitude of the vertical force arising from basal interactions can be written compactly
 75 as

$$\int_a \frac{\sigma_n}{\sqrt{\left(\frac{\partial b}{\partial x} \right)^2 + \left(\frac{\partial b}{\partial y} \right)^2 + 1}} da = \int_{\mathcal{A}} \sigma_n \, d\mathcal{A} . \quad (\text{S3})$$

76 The liquid pressure and the ice normal stress are equal at the ice–liquid interface over
 77 macroscopic drainage elements. Away from such drainage elements, ice–mineral interac-
 78 tions generate the thermomolecular pressure that supports the portion of the overburden
 79 that is not balanced by liquid pressure. A second distinct source of differences between
 80 σ_n and P arises from the effects of surface energy on curved ice–liquid interfaces that is

81 known as the Gibbs–Thomson effect. The Gibbs–Thomson effect does not have a direct
 82 role in supporting the glacier weight, but is nevertheless important for maintaining the
 83 ice–liquid equilibrium temperature at a uniform level over mm-scale distances between
 84 roughness elements, for example in the pore throats between particles when the glacier
 85 bed is comprised of unconsolidated sediment. On an ice–liquid interface with unit normal
 86 $\hat{\mathbf{n}}$ directed outwards from the ice, the interfacial curvature is $\nabla \cdot \hat{\mathbf{n}}$ and the combined effects
 87 of the Gibbs–Thomson effect and the local thermomolecular pressure $p_T(\epsilon)$ give rise to an
 88 ice–liquid pressure difference of (e.g. Rempel, 2008, Eq. 2)

$$P_i - P \approx \sigma_n - P = \gamma_{il} \nabla \cdot \hat{\mathbf{n}} + p_T(\epsilon) ,$$

89 where $\gamma_{il} \approx 0.03 \text{ J m}^{-2}$ is the ice–liquid surface energy and ϵ is the local thickness of the
 90 premelted liquid films that separate the ice from the mineral substrate. Mathematically, the
 91 integral of curvature over a surface can be expressed as a line integral along its boundary,
 92 summing up components that are tangent to the surface along that boundary (e.g. Rempel
 93 and others, 2001). Hence, when integrated over an area along the basal surface that
 94 is chosen so that its tangent vector has only horizontal components on its boundaries
 95 (e.g. following the trace of the surface along the midpoints of pore throats), the net vertical
 96 force resulting from the thermomolecular pressure is exactly

$$\int_a \frac{p_T}{\sqrt{\left(\frac{\partial b}{\partial x}\right)^2 + \left(\frac{\partial b}{\partial y}\right)^2 + 1}} da = \int_a \frac{\sigma_n - P}{\sqrt{\left(\frac{\partial b}{\partial x}\right)^2 + \left(\frac{\partial b}{\partial y}\right)^2 + 1}} da ;$$

97 the Gibbs–Thomson effect produces zero contribution to the integral on the right side. This
 98 is a case where rigorous mathematical description may prove less satisfying than simple
 99 physical reasoning. Demanding that surface–energy produce a net upwards force is akin
 100 to asking that the ice somehow be supported by a surface traction that has no means of
 101 exerting a net force against anything other than the adjacent ice on its boundaries (i.e. the
 102 glaciological equivalent of “pulling itself up by its bootstraps”).

103 In the main text we define $P_T = \sigma_n - P$ as the homogenized thermomolecular pressure
 104 that is the net force per unit macroscopic basal area arising from ice–mineral interactions;
 105 while P_T is suitably defined and may be regarded as uniform over macroscopic areas, over
 106 length scales of millimeters or smaller, the local thermomolecular pressure $p_T(\epsilon)$ that is
 107 responsible for generating this net pressure is itself made heterogeneous by local variations
 108 in ice–mineral separation ϵ and interfacial curvature $\nabla \cdot \hat{\mathbf{n}}$. (Further discussion of other
 109 glaciological consequences of such behavior is contained in Rempel, 2008; Meyer and others,
 110 2018, 2019). Substituting the definition for P_T allows the integral of σ_n to be expressed as

$$\int_{\mathcal{A}} \sigma_n d\mathcal{A} = \int_{\mathcal{A}} (P + P_T) d\mathcal{A} . \quad (\text{S4})$$

111 The vertical force balance is obtained by equating equation (S2) and (S4) to yield equation
 112 (3) in the main text.

113 To arrive at equation (4) in the main text, we define the average ice-equivalent thickness
 114 as

$$H_0 = \frac{1}{\mathcal{A}} \int_{\mathcal{A}} H \, d\mathcal{A},$$

115 so that with the average liquid pressure defined as

$$\bar{P} = \frac{1}{\mathcal{A}} \int_{\mathcal{A}} P \, d\mathcal{A},$$

116 and the average thermomolecular pressure over the portion of the bed area that does not
 117 contain macroscopic drainage elements defined as

$$\bar{P}_T = \frac{1}{(1-\phi)\mathcal{A}} \int_{(1-\phi)\mathcal{A}} P_T \, d\mathcal{A},$$

118 the leftmost and rightmost sides of equation (3) can be expressed as

$$\rho_i g H_0 \mathcal{A} = \bar{P} \mathcal{A} + (1-\phi) \bar{P}_T \mathcal{A}. \quad (\text{S5})$$

119 For the idealized case considered, with a homogeneous liquid pressure $P = \bar{P}$ and with
 120 $P_T = \bar{P}_T$, equation (S5) simplifies to equation (4) in the main text.

121 Variations in Equilibrium Temperature – Equations (5)–(7)

122 The expressions in equations (5)–(7) of the main text are obtained by simple substitution
 123 into equation (1). It is worth emphasizing the differences in scale of changes in equilibrium
 124 temperature with changes in the controlling liquid pressure and ice normal stress. In
 125 particular, we have that at constant values of $P_T = \sigma_n - P$ (e.g. along the ice-walled
 126 boundaries of macroscopic drainage elements, with $P_T = 0$),

$$\left. \frac{\partial T_{\text{eq}}}{\partial P} \right|_{\text{constant } P_T} = -C_0.$$

127 This can be contrasted with the behavior that results from variations in P away from
 128 drainage elements, where P_T must also change to maintain the vertical force balance over
 129 representative projected areas \mathcal{A} that is described by equation (4) of the main text (or S5
 130 above), so that

$$\left. \frac{\partial T_{\text{eq}}}{\partial P} \right|_{P_T=(P_0-P)/(1-\phi)} = \frac{C_0}{1-\phi} \frac{\rho_i + \phi(\rho_l - \rho_i)}{\rho_l - \rho_i}.$$

131 Because the density difference between the phases is about 10% of the densities of either
 132 the liquid or the solid, the equilibrium temperature over macroscopic drainage elements,
 133 where $P_T = 0$, tends to be about an order of magnitude less sensitive to liquid pressure
 134 changes than the equilibrium temperature over premelted regions.

135 **Perturbed Temperature Profiles and Surface Gradients – Equa-**
 136 **tions (8) and (13)**

137 Equation (8) in the main text gives the perturbation to the temperature gradient along
 138 the sliding interface following a jump in equilibrium temperature of size ΔT ; this result
 139 can be found in standard reference texts, including Carslaw and Jaeger (1959). To further
 140 explore how the temperature profile responds to such a change in boundary temperature,
 141 consider the case in which the temperature profile within the ice immediately prior to the
 142 jump in temperature is steady at $\tilde{T}(z, 0) = \tilde{T}_i(z)$. We define the perturbed temperature
 143 as $T(z, t) = \tilde{T}(z, t) - \tilde{T}_i(z)$ so that when the heat transport can be considered to follow
 144 Fourier’s law with a constant thermal conductivity, the evolution of T satisfies

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2},$$

145 where κ is the thermal diffusivity. Introducing the similarity variable $\eta = z/(2\sqrt{\kappa t})$, this
 146 can be written as

$$\frac{d^2 T}{d\eta^2} = -2\eta \frac{dT}{d\eta}.$$

147 Integrating twice and applying the boundary conditions that $T(0, t) = \Delta T$ and $T(\infty, t) = 0$
 148 (with initial condition $T(z > 0, 0) = 0$) gives the perturbed temperature field as

$$T(z, t) = \Delta T \operatorname{erfc} \left(\frac{z}{2\sqrt{\kappa t}} \right),$$

149 and the perturbation to the temperature gradient as

$$\frac{\partial T}{\partial z} = -\frac{\Delta T}{\sqrt{\pi \kappa t}} \exp \left(\frac{-z^2}{4\kappa t} \right).$$

150 Equation (8) in the main text expresses the perturbed temperature gradient at $z = 0$.

151 The analysis is easily extended to consider the response of the temperature field to
 152 any number of subsequent changes in the boundary temperature at $z = 0$. For example,
 153 consider the case where, after applying a jump of ΔT at $t = 0$, the boundary temperature
 154 subsequently changes discontinuously again by $-\Delta T$ at $t = t_1$. The perturbed temperature
 155 field that we just determined satisfies the heat equation, so we can define $T_1(z, t \geq t_1) =$
 156 $\tilde{T}(z, t \geq t_1) - T(z, t \geq t_1)$, and look for a similarity solution in the same manner, finding
 157 that

$$T_1(z, t \geq t_1) = -\Delta T \operatorname{erfc} \left[\frac{z}{2\sqrt{\kappa(t - t_1)}} \right].$$

158 Hence the temperature field itself for $t \geq t_1$ is

$$\tilde{T}(z, t \geq t_1) = \tilde{T}_i + \Delta T \operatorname{erfc} \left(\frac{z}{2\sqrt{\kappa t}} \right) - \Delta T \operatorname{erfc} \left[\frac{z}{2\sqrt{\kappa(t - t_1)}} \right],$$

159 with gradient

$$\frac{\partial \tilde{T}}{\partial z} = -\frac{\Delta T}{\sqrt{\pi \kappa t}} \exp\left(\frac{-z^2}{4\kappa t}\right) + \frac{\Delta T}{\sqrt{\pi \kappa t}} \exp\left[\frac{-z^2}{4\kappa(t-t_1)}\right] + \frac{\partial \tilde{T}_i}{\partial z}.$$

160 For ice sliding at a fixed velocity u_s , the time taken to reach location x is x/u_s and the
161 time elapsed since reaching a cavity's termination at $x = \ell$ is $t - t_1 = (x - \ell)/u_s$; making
162 these two substitutions, the perturbation to the temperature gradient at $z = 0$ is given by
163 equation (13) in the main text.

164 Extension For High Englacial Liquid Concentrations

165 The Gibbs–Thomson effect, along with minor quantities of soluble impurities, enables liquid
166 veins to persist along the triple junctions where three ice grains meet, connecting at nodes
167 to form a liquid network that is in equilibrium with temperate ice (e.g. Nye and Frank, 1973).
168 Estimates of the upper bound on liquid fraction at Blue Glacier by Raymond and Harrison
169 (1975, Table III) suggest that values as high as 0.09% might have been present in fine-
170 grained ice both near the surface and near the bed, and the question arises as to the effects
171 of such liquid contents on the heat transport that enables basal freeze-on. To ascertain the
172 likely importance of this complication, we first recognize that the liquid volume fraction
173 itself n_l is expected to be a decreasing function of the temperature departure from bulk
174 equilibrium (i.e. with $P = \sigma_n$). Including the effect of changes in n_l in the energy balance
175 (while neglecting advective transport) leads to

$$\frac{\partial T}{\partial t} \left(1 - \frac{\mathcal{L}}{C_p} \frac{dn_l}{dT}\right) = \kappa \frac{\partial^2 T}{\partial z^2},$$

176 where \mathcal{L} is the latent heat and C_p is the specific heat capacity. Since englacial liquid
177 contents are expected to become appreciable only in temperate ice, we anticipate that
178 dn_l/dT should produce a significant contribution to the left side of the equation over a
179 fairly limited temperature range.

180 Although the precise form of the dependence of n_l on T is likely to depend on a number
181 of highly variable factors, such as the ice crystal size distribution and typical aspect ratios,
182 we can make progress by considering limiting cases. For example, if the liquid content owes
183 its presence primarily to the colligative effects of minor quantities of impurities, we might
184 expect that n_l should be well approximated over a limited range by a linear function of
185 temperature. In this case, the results of the analysis above and in the main text remain
186 functionally the same, but with a larger modified effective thermal diffusivity $\hat{\kappa}$ replacing
187 its dry value as $\hat{\kappa} = \kappa/[1 - (\mathcal{L}/C)(dn_l/dT)]$. For example, this would be expected to
188 reduce the thickness of freeze-on that accompanies transport over a single cavity from the

189 expression given in equation (9) of the main text to

$$h_0 \approx \frac{2}{\sqrt{\pi}} \ell \frac{C_p \Delta T}{\mathcal{L}} \sqrt{\frac{\kappa}{u_s \ell \left(1 - \frac{\mathcal{L}}{C_p} \frac{dn_l}{dT}\right)}}.$$

190 For intuition, given a temperature change of ΔT , this modified formula produces a predic-
191 tion for h_0 that is noticeably smaller than that from equation (9) of the main text if the
192 change Δn_l is comparable to or larger than $C_p \Delta T / \mathcal{L}$, or about 0.06 % when $\Delta T = 0.1$ °C.
193 The reports from Blue Glacier suggest that such complications may be important in some
194 portions of at least some temperate glaciers, but the prevalence of such conditions is not
195 well constrained.

196 We note that in circumstances where the impurity content in the ice is sufficiently small
197 that the liquid fraction is controlled primarily by the Gibbs–Thomson effect rather than
198 by the temperature depression associated with colligative effects, the change in n_l with T
199 might be expected to be nonlinear (e.g. Rempel, 2005), causing the sensitivity of diffusive
200 heat transport to latent heat effects to decrease as the temperature cools. Moreover, at
201 large values of n_l , the permeability of the ice to liquid transport may further affect the
202 temperature field. As there is currently only very sparse quantitative data available to
203 constrain the absolute levels of n_l in temperate basal ice, let alone changes in n_l with T ,
204 we leave further speculation over such potential effects to future work.

205 References

- 206 Andrews LC, Catania GA, Hoffman MJ, Gulley JD, Lüthi MP, Ryser C, Hawley RL and
207 Neumann TA (2014) Direct observations of evolving subglacial drainage beneath the
208 Greenland Ice Sheet. *Nature*, **514**(7520), 80–83
- 209 Carslaw HS and Jaeger JC (1959) *Conduction of Heat in Solids*. Oxford: Clarendon Press,
210 2nd ed.
- 211 Huss M, Bauder A, Werder M, Funk M and Hock R (2007) Glacier-dammed lake outburst
212 events of Gornensee, Switzerland. *Journal of Glaciology*, **53**(181), 189–200
- 213 Kamb WB (1961) The thermodynamic theory of nonhydrostatically stressed solids. *Journal*
214 *of Geophysical Research*, **66**(1), 259–271
- 215 Meyer CR, Downey AS and Rempel AW (2018) Freeze-on limits bed strength beneath
216 sliding glaciers. *Nat. Commun.*, **9**(1), 3242 (doi: 10.1038/s41467-018-05716-1)
- 217 Meyer CR, Robel AA and Rempel AW (2019) Frozen fringe explains sediment freeze-on
218 during Heinrich events. *Earth and Planetary Science Letters*, **524**, 115725
- 219 Nye J and Frank F (1973) Hydrology of the intergranular veins in a temperate glacier. In
220 *Symposium on the Hydrology of Glaciers*, volume 95, 157–161, Citeseer
- 221 Nye JF (1953) The flow law of ice from measurements in glacier tunnels, laboratory ex-

- 222 periments and the Jungfraufirn borehole experiment. *Proc. R. Soc. Lond. Ser. A* 219,
223 477–489 (doi: 10.1098/rspa.1953.0161)
- 224 Paterson M (1973) Nonhydrostatic thermodynamics and its geologic applications. *Reviews*
225 *of Geophysics*, **11**(2), 355–389
- 226 Raymond C and Harrison W (1975) Some observations on the behavior of the liquid and
227 gas phases in temperate glacier ice. *Journal of Glaciology*, **14**(71), 213–233
- 228 Rempel A (2005) Englacial phase changes and intergranular flow above subglacial lakes.
229 *Annals of Glaciology*, **40**, 191–194
- 230 Rempel AW (2008) A theory for ice-till interactions and sediment entrainment beneath
231 glaciers. *J. Geophys. Res.*, **113**(F1), ISSN 2156-2202 (doi: 10.1029/2007JF000870),
232 f01013
- 233 Rempel AW, Wettlaufer JS and Worster MG (2001) Interfacial premelting and the ther-
234 momolecular force: Thermodynamic buoyancy. *Phys. Rev. Lett.*, **87**(8), 088501 (doi:
235 10.1103/PhysRevLett.87.088501)
- 236 Röthlisberger H (1972) Water pressure in intra-and subglacial channels. *Journal of Glaciol-*
237 *ogy*, **11**(62), 177–203
- 238 Sekerka RF and Cahn JW (2004) Solid–liquid equilibrium for non-hydrostatic stress. *Acta*
239 *Materialia*, **52**(6), 1663–1668