Statistics 410, Fall 2011

Solutions to homework 4, due 10/19/11

4.16 U is the dosage level (1 through 7). Each level has probability 1/7, so the histogram is like Figure 4.7 except height 1/7 and with 7 bars instead of 6.

b.

$$\mu = E(X) = \frac{1}{7}1 + \frac{1}{7}2 + \cdots + \frac{1}{7}7 = \frac{1}{7}28 = 4.$$

$$Var(X) = \frac{1}{7}(1-4)^2+\frac{1}{7}(2-4)^2+\cdots+\frac{1}{7}(7-4)^2 = \frac{1}{7}(9+4+1+0+1+4+9) = 4$$

and so $\sigma = 2$.

c.

$$P(\mu - \sigma \le U \le \mu + \sigma) = P(2 \le U \le 6).$$

This is $f(2) + f(3) + \cdots + f(6) = 5/7$.

4.22 (a) There are 6^3 outcomes. 6 of those have all three rolls show the same face. So $f(1) = 6/6^3 = 1/36$.
To count how many outcomes have all three rolls showing different faces, realize that the first roll can be anything. Then the second roll must be different from the first, which has probability 5/6. Then, assuming the first two rolls were different, the third must be different from the first two, which has probability 4/6. So $f(3) = (5/6)(4/6) = 20/36$.
Then $f(2) = 1 - f(1) - f(3) = 15/36$.

(b) $F(1) = 1/36$. $F(2) = f(1) + f(2) = 16/36 = 4/9$. $F(3) = 1$.

(c)

$$\mu = \frac{1}{36}1 + \frac{15}{36}2 + \frac{20}{36}3 = \frac{91}{36} = 2.5278.$$

$$E(X^2) = \frac{1}{36}1 + \frac{15}{36}4 + \frac{20}{36}9 = \frac{241}{36} = 6.6944.$$

So $Var(X) = E(X^2) - E(X)^2 = 6.6944 - 6.3898 = .3046$. So $\mu = .5519$.

4.23 (a) K can be $0, 1$ or 2. $f(0) = (6/8) \cdot (5/7) = 30/56 = 15/28$. $f(2) = (2/8)(1/7) = 2/56 = 1/28$. It follows that

$$f(1) = 1 - f(0) - f(2) = 12/28 = 3/7$$

(b)

$$F(0) = 15/28, F(1) = 27/28, F(2) = 1.$$

(c)

$$\mu = E(K) = \frac{15}{28}0 + \frac{3}{7}1 + \frac{1}{28}2 = \frac{14}{28} = .5$$

(d)

$$E((K-.5)^2) = \frac{15}{28}\frac{1}{4} + \frac{3}{7}\frac{1}{4} + \frac{1}{28}\frac{9}{4} = \frac{27}{112} \approx .2411.$$

So $\sigma = .4910$.

4.30 (a) There are 36 possibilities. 11 of them have 1 as the lowest die. 9 have 2 as the lowest die. 7 have 3 as the lowest die. 5 have 4 as the lowest die. 3 have 5 as the lowest die and 1 has 6 as the lowest die.

x	$f(x)$
1	$f(1) = \frac{11}{36}$
2	$f(2) = \frac{9}{36}$
3	$f(3) = \frac{7}{36}$
4	$f(4) = \frac{5}{36}$
5	$f(5) = \frac{3}{36}$
1	$f(1) = \frac{1}{36}$

(b)

$$E(X) = \frac{11}{36}1 + \frac{9}{36}2 + \frac{7}{36}3 + \frac{5}{36}4 + \frac{3}{36}5 + \frac{1}{36}6 = \frac{91}{36} \approx 2.5278.$$

(c)

$$E(X^2) = \frac{11}{36}1 + \frac{9}{36}4 + \frac{7}{36}9 + \frac{5}{36}16 + \frac{3}{36}25 + \frac{1}{36}36 = \frac{301}{36} \approx 8.3611.$$

So $Var(X) = 8.3611 - 6.3898 = 1.9713$. So $\sigma \approx 1.4040$.

4.31 There are b^2 possibilities for the two dice. There are $2n-1$ ways to have 1 be the lowest number showing. There are $2n-2k+1$ ways to have k be the lowest number showing. So

$$f(k) = \frac{2n-2k+1}{n^2}.$$

As a reality check, we need $\sum_{k=1}^{n} f(k) = 1$. This is equivalent to

$$\sum_{k=1}^{n}(2n-2k+1) = 2n^2 - (n(n+1)) + n = 2n^2 - n^2 - n + n = n^2.$$

(Embedded in that equation, we used $\sum_{k=1}^{n} k = n(n+1)/2$.)

$$E(X) = \sum_{k=1}^{n} \frac{2n-2k+1}{n^2} k = \frac{1}{n^2}[n^2(n+1) - n(n+1)(2n+1)/3 + n(n+1)/2] =$$

$$\frac{1}{n}[n(n+1) - (n+1)(2n+1)/3 + (n+1)/2] = \frac{1}{n}[\frac{1}{3}n^2 + \frac{n}{2} + \frac{1}{6}] = \frac{n}{3} + \frac{1}{2} + \frac{1}{6n}$$

For the calculation above we needed both the sum we used in the preceding paragraph and also $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$.

To find the standard deviation, we start by calculation

$$E(X^2) = \sum_{k=1}^{n} \frac{2n-2k+1}{n^2} k^2 = \frac{2}{n} \frac{n(n+1)(2n+1)}{6} - \frac{2}{n^2} \frac{n(n+1)}{2}^2 + \frac{1}{n^2} \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{3} - \frac{(n+1)^2}{2} + \frac{(n+1)(2n+1)}{6n} = \frac{4n^2+6n+2}{6} - \frac{3n^2+6n+3}{6} + \frac{2n^2+3n+1}{6n}$$

$$= \frac{n^3-n}{6n} + \frac{2n^2+3n+1}{6n} = \frac{n^3+2n^2+2n+1}{6n} = \frac{(n+1)(n^2+n+1)}{6n}$$

So

$$Var(X) = \frac{6n(n^3+2n^2+2n+1)}{(6n)^2} - (\frac{2n^2+3n+1}{6n})^2 = \frac{2n^4-n^2-1}{(6n)^2}.$$

So the standard deviation is

$$\sigma = \frac{\sqrt{2n^4-n^2-1}}{6n}.$$

4.32 (a) We first note that $P(X=0) = .5$ When $X=2$, that means we got (reading left to right) TTH. The probability of this is $.5^3$. When $X=4$, we must have gotten $TTTTH$, which has probability $.5^5$. So

$$P(X \text{ even }) = .5 + .5^3 + .5^5 + \cdots = .5(1 + r + r^2 + r^3 + \cdots)$$

where $r = .25 = .5^2$. The infinite sum $1 + r + r^2 + \cdots = \frac{1}{1-r} = \frac{4}{3}$. So the probability is 2/3.

(b) Since X is either even or odd, we see that $P(X \text{ odd }) = 1 - \frac{2}{3} = 1/3$. We could repeat a similar argument to the first case also if we wanted to.

5.1 Here $\pi = 1/13$. So $\mu = \pi = 1/13$.

$$\sigma = \sqrt{\pi(1-\pi)} = \sqrt{\frac{1}{13}\frac{12}{13}} = \frac{\sqrt{12}}{13}.$$

5.2 This is Bernoulli with $\pi = 3/4$. So $\mu = .75$, $\sigma = \sqrt{\frac{3}{4}\frac{1}{4}} = \frac{\sqrt{3}}{2}$.

5.5 We have a Bernoulli variable, so we know $\mu = \pi$. We also know that the only possible values of X are 0 and 1. So

$$\begin{aligned} E((X-\mu)^3) &= \pi(1-\pi)^3 + (1-\pi)(-\pi)^3 = \pi - 3\pi^2 + 3\pi^3 - \pi^4 + (-\pi^3) + \pi^4 \\ &= \pi - 3\pi^2 + 2\pi^3 = \pi(1 - 3\pi + 2\pi^2) = \pi(1-\pi)(1-2\pi). \end{aligned}$$

So the coefficient of skewness is

$$\frac{\pi(1-\pi)(1-2\pi)}{[\pi(1-\pi)]^{3/2}} = \frac{1-2\pi}{\sqrt{\pi(1-\pi)}}.$$

5.7

5.8 (a) $\mu = n\pi = 40$. $\sigma = \sqrt{n\pi(1-\pi)} = \sqrt{100 \cdot .4 \cdot .6} = \sqrt{24} \approx 4.899$.

(b) $\mu = n\pi = 160$. $\sigma = \sqrt{n\pi(1-\pi)} = \sqrt{400 \cdot .4 \cdot .6} = \sqrt{96} \approx 9.798$.

(c) $\mu = n\pi = 60$. $\sigma = \sqrt{n\pi(1-\pi)} = \sqrt{100 \cdot .4 \cdot .6} = \sqrt{24} \approx 4.899$.

(d) $\mu = n\pi = 600$. $\sigma = \sqrt{n\pi(1-\pi)} = \sqrt{400 \cdot .4 \cdot .6} = \sqrt{96} \approx 9.798$.

5.10 If we let 1 be the outcome of a correct guess, and 0 of an incorrect guess, we get a Bernoulli R.V. with $\pi = 1/3$. 12 guesses then corresponds to a binomial R.V., K, with $n = 12$, $\pi = 1/3$.

(a) $f_K(4) = \binom{12}{4}(\frac{1}{3})^4(\frac{2}{3})^8 \approx .2384$.

(b) $f_K(5) = \binom{12}{5}(\frac{1}{3})^5(\frac{2}{3})^7 \approx .1908$.

(c) $f_K(6) = \binom{12}{6}(\frac{1}{3})^6(\frac{2}{3})^6 \approx .1113$.

(d) This is

$$\sum_{k=1}^{5} \binom{12}{k} (\frac{1}{3})^k (\frac{2}{3})^{12-k}$$

which is (by computation), .8146.

(e) This is 1 minus the previous result, so .1854.

(f) This is

$$\sum_{k=7}^{12} \binom{12}{k} (\frac{1}{3})^k (\frac{2}{3})^{12-k}$$

which is .0664.

5.11 We consider survival of the Ebola infection as Bernoulli with $\pi = .1$. Here the result 1 corresponds to survival, and 0 corresponds to non-survival. We consider the 10 independent cases to be a binomial random variable K with $\pi = .1$, $n = 10$.

(a) $f_K(0) = \binom{10}{0}(.9)^{10} \approx .3487$.

(b) $f_K(1) = \binom{10}{1}(.1)(.9)^9 \approx .3874$.

(c) $f_K(2) = \binom{10}{2}(.1)^2(.8)^8 \approx .1937$.

(d) This is the sum of the first two probabilities, which is .7361.

(e) This is 1 minus the previous result: .2639.

(f) This is 1 minus the sum of the first three results: .07019.

5.14 Each question is a Bernoulli R.V. with a 1 for a correct answer, and a 0 for an incorrect answer, and $\pi = .25$. We do 5 trials, which gives K, a binomial R.V. with $\pi = .25$, $n = 5$.

(a) $f_K(1) = \binom{5}{1}(.25)^1(.75)^4 = .3955$.

(b) $f_K(3) = \binom{5}{3}(.25)^3(.75)^2 = .0879$.

(c) We're looking for $f_k(3) + f_K(4) + f_K(5)$. This is

$$\binom{5}{3}(.25)^3(.75)^2 + \binom{5}{4}(.25)^4(.75)^1 + \binom{5}{5}(.25)^5(.75)^0 = .1035.$$

(d) The probability of getting them all right is $.25^5 = .00098$. So the probability of at least one wrong is $1 - .00098 = .99902$.

5.21 We treat the chance of an infected person dying as a Bernoulli R.V. with a 1 if the subject dies, and a 0 if not. $\pi = 63/877 \approx .07184$. If we examine 10 independent cases, we represent this as a binomial R. V. with $n = 10$, $\pi = 63/877$.

(a) $f_K(0) = (\frac{814}{877})^{10} = .4745$.

(b) $f_K(2) = \binom{10}{2}(\frac{63}{877})^2(\frac{814}{877})^8 = .1279$.

(c) This is $f_K(0) + f_K(1) + f_K(2)$. $f_K(1) = .9697$.

5.22 As in the previous problem, we treat an infected person dying as a Bernoulli R.V. with a 1 if the subject dies and a 0 otherwise. $\pi = 3/70 \approx .04286$. We treat the outcome (total number of deaths) of 10 independent cases as a binomial R. V. with $\pi = 3/70$ and $n = 10$.

(a) $f_K(0) = (67/70)^{10} = .6453$.

(b) $f_K(1) = 10 \cdot (3/70) \cdot (67/70)^9 \approx .2889$.

(c) This is the sum of the probabilities above. .9343.

5.36 We did 5.34 in class by a slightly different means.

$$E(K(K-1)(K-2)) = \sum_{k=0}^{n} \binom{n}{k} [k(k-1)(k-2)]\pi^k(1-\pi)^{n-k} =$$

$$\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}[k(k-1)(k-2)]\pi^k(1-\pi)^{n-k} =$$

$$\pi^3 n(n-1)(n-2) \sum_{k=3}^{n} \frac{(n-3)!}{(k-3)!(n-k)!}\pi^{k-3}(1-\pi)^{n-k} =$$

$$\pi^3 n(n-1)(n-2) \sum_{k=0}^{n-3} \binom{n-3}{k} \pi^k(1-\pi^k) = \pi^3 n(n-1)(n-2)(\pi+(1-\pi))^{n-3} = \pi^3 n(n-1)(n-2).$$

Then we get

$$\pi^3 n(n-1)(n-2) = E(K(K-1)(K-2)) = E(K^3) - 3E(K^2) + 2E(K) = E(K^3) - 3[\pi^2(n^2-n) + n\pi] + 2n\pi$$

so $E(K^3) = \pi^3 n(n-1)(n-2) + \pi^2 3n(n-1) + \pi n$.

So, using $\mu = n\pi$

$$E((K-\mu)^3) = E(K^3) - 3n\pi E(K^2) + 3n^2\pi^2(n\pi) - n^3\pi^3 =$$

$$\pi^3 n(n-1)(n-2) + \pi^2 3n(n-1) + \pi n - 3n[\pi^3 n(n-1) + \pi^2 n] + 2\pi^3 n^3$$

$$= \pi^3[2n] + \pi^2[-3n] + \pi n = n\pi(\pi-1)(2\pi-1)$$

Finally,

$$\frac{E((K-\mu)^3)}{\sigma^3} = \frac{n\pi(\pi-1)(2\pi-1)}{(n\pi(1-\pi))\sqrt{n\pi(1-\pi)}} = \frac{2\pi-1}{\sqrt{n\pi(1-\pi)}}.$$