# Spectral Sequence Notes: Filtered algebras, Feb. 17.

## 1. The Lie filtration of the Universal Enveloping Algebra

We start with an example.

**Definition 1.1.** *Let $V$ be a vector space over a field $k$. We define a unital associative algebra*

$$TV := k \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \cdots = \bigoplus_{i=0}^{\infty} V^{\otimes i}.$$

*Multiplication is by concatenation so that*

$$(a_1 \otimes \cdots \otimes a_k) \otimes (b_1 \otimes \cdots \otimes b_l) = (a_1 \otimes \cdots \otimes a_k \otimes b_1 \otimes \cdots \otimes b_l)$$

**Definition 1.2.** *Let $L$ be a Lie algebra over a field $k$. Let $I$ be the two-sided ideal of $TL$ generated by all elements of the form $[x, y] - x \otimes y + y \otimes x$ for $x, y \in L$.*

$$U(L) = TL/I$$

The algebra $U(L)$ has a number of nice properties.

1. The ideal is generated by the difference between the commutator of $x$ and $y$ in $TV$ and the Lie bracket of $x$ and $y$. Thus $U(L)$ contains $L$ as a sub-Lie algebra (where the bracket in $U(L)$ is the commutator).
2. If $L$ is a Lie algebra with bracket equal to zero, then $U(L)$ is the symmetric algebra on $L$. That is, given a vector space basis of $L$, $\{x_1, x_2, \dots\}$,
$$U(L) = k[x_1, x_2, \dots].$$
3. In general, $U(L)$ is the size of the symmetric algebra on $L$. That is, choose an ordered basis for $L$, $\{x_1, x_2, \dots\}$. Then $U(L)$ has a basis of 1 together with monomials $x_{i_1}^{r_1} x_{i_2}^{r_2} \cdots x_{i_k}^{r_k}$ where $i_1 < i_2 < \cdots < i_k$ and $r_i \in \mathbf{Z}_{>0}$.
   This is easy to prove by using the relations in $I$ to take any monomial of length $n$ in the basis elements, and reorder to get a monomial of length $n$ with the right order plus monomials of length less than $n$.

**Definition 1.3.** *Let $A$ be a $k$-algebra. An* algebra filtration *of $A$ is either*

1. *an* increasing filtration*: $0 = F^{-1}A \subseteq F^0 A \subseteq F^1 A \subseteq \cdots$ satisfying*
   (a) *The unit map $k \to A$ factors through $F^0 A$ and*
   (b) $(F^p A) \cdot (F^q A) \subseteq F^{p+q} A$.
2. *a* decreasing filtration*: $A = F^0 A \supseteq F^1 A \supseteq \cdots$ satisfying*
   (a) $F^1 A \subseteq F^0 A = A \xrightarrow{\epsilon} k$ *is* $0$.
   (b) $(F^p A) \cdot (F^q A) \subseteq F^{p+q} A$.

*The associated graded $E^0 A$ is, in the first case $\oplus_{p=0}^{\infty} F^p A / F^{p-1} A$ and in the second case $\oplus_{p=0}^{\infty} F^p A / F^{p+1} A$. In both cases $E^0 A$ is a graded, augmented $k$-algebra.*

**Definition 1.4.** *The algebra $TV$ can be graded by length of monomials. $U(L)$ cannot be graded this way since the generators of $I$ are not homogeneous with respect to length of monomials. But the grading on $TV$ gives a filtration of $U(L)$ called the* Lie filtration.

$$F^p U(L) = image(k \oplus L \oplus L^{\otimes 2} \oplus \cdots \oplus L^{\oplus p})$$

*That is $F^p U(L)$ is the image of those tensors of length $\leq p$ from $TL$.*

The algebra $TL$ can be graded by using the tensor length. That grading induces a filtration on $U(L)$.

**Lemma 1.5.**
$$E^0U(L) \cong \mathit{Sym}(L)$$

The right hand side is the "symmetric algebra" on the underlying vector space of $L$. That is, if $\{x_1, x_2, \dots\}$ is a vector space basis for $L$, then
$$\mathrm{Sym}(L) = k[x_1, x_2, \dots].$$
We leave the proof as an exercise, but we observe that multiplication of elements from $L$ is now commutative. If $x, y \in L$, then in $U(L)$,
$$x \otimes y - y \otimes x = [x, y], \text{ so } x \otimes y = y \otimes x + [x, y].$$
In $F^2U(L)/F^1U(L)$, $[x, y] = 0$, so $\overline{x \otimes y} = \overline{y \otimes x}$.

## 2. The bar resolution and bar complex

The bar resolution gives a natural resolution of $k$ by free left $A$-modules. Of course we can calcuate Ext and Tor with any resolution, but to develop our spectral sequences we want a situation where a filtration of our algebra leads to a filtration of the resolution.

Let $A$ be a unital, associative, augmented $k$-algebra.

**Definition 2.1.** *The* bar resolution *of $k$, $(B_*(A), d_*)$ is given by*

(1) $B_n(A) = A^{\otimes(n+1)} := A \otimes_k A \otimes_k A \otimes_k \cdots \otimes_k A.$

(2) *We write as shorthand*
$$a_0[a_1|\cdots|a_n] \textit{ for } a_0 \otimes a_1 \otimes \cdots \otimes a_n \in B_n(A).$$

(3) $d_0(a) = \epsilon(a)$ *For $n > 0$,*
$$d_n(a_0[a_1|\cdots|a_n]) = \sum_{i=0}^{n-1}(-1)^i a_0[a_1|\cdots|a_ia_{i+1}|\cdots|a_n] + (-1)^n a_0[a_1|\cdots|a_{n-1}]\epsilon(a_n).$$

The notation is chosen to be compact, and to emphasize the left $A$-module structure by putting the leftmost copy of $A$ on the outside of the brackets. It also emphasizes that $B_n(A)$ is a free $A$-module, with an $A$-basis given by taking a vector space basis inside the brackets.

**Lemma 2.2.** *$d_*$ is a differential, that is $d_{n-1} \circ d_n = 0$.*

This is a standard proof like many similar proofs in first year algebraic topology.

**Lemma 2.3.** *$(B_*A, d_*)$ is a free resolution of $k$ by left $A$-modules. That is the augmented bar resolution,*
$$\cdots \xrightarrow{d_3} A \otimes_k A \otimes_k A \xrightarrow{d_2} A \otimes_k A \xrightarrow{d_1} A \xrightarrow{d_0} k$$
*is exact.*

*Proof.* We prove this by constructing a contracting homotopy, $s_0$.
$$s_0(a_0[a_1|\cdots|a_n]) = 1[a_0|\cdots|a_n].$$
We want to show
$$d_{n+1}s_0 + s_0d_n = 1_{B_n(A)}$$

which shows that the identity map on the augmented bar resolution is chain homotopic to the zero map, and thus that there is no homology.

We first consider the exceptional case $n = 0$.

$$(d_1 s_0 + s_0 d_0)(a) = d_1(1[a]) + s_0\epsilon(a) = a - 1 \cdot \epsilon(a) + 1 \cdot \epsilon(a) = a.$$

Now we do the general case:

$$\begin{aligned}(d_{n+1}s_0 + s_0 d_n)(a_0[a_1|\cdots|a_n] = d_{n+1}(1[a_0|\cdots|a_n]) + s_0(\sum_{i=0}^{n-1}(-1)^i a_0[a_1|\cdots|a_i a_{i+1}|\cdots|a_n] + \\ (-1)^n a_0[a_1|\cdots|a_{n-1}]\epsilon(a_n)) \\ =& a_0[a_1|\cdots|a_n] + \sum_{i=1}^{n}(-1)^i 1[a_0|\cdots|a_{i-1}a_i|\cdots|a_n] + (-1)^{n+1}1[a_0|\cdots|a_{n-1}]\epsilon(a_n) \\ &+ \sum_{i=0}^{n-1}(-1)^i 1[a_0|\cdots|a_i a_{i+1}|\cdots|a_n] + (-1)^n 1[a_0|\cdots|a_{n-1}]\epsilon(a_n)\end{aligned}$$

Clearly all terms cancel other than $a_0[a_1|\cdots|a_n]$. So we've established that $(d_{n+1}s_0 + s_0 d_n) = 1_{B_n(A)}$. ∎

2.1. **The bar complex and Tor.** Recall that for a left $A$-module $N$ and a right $A$-module $M$, $\mathrm{Tor}^A(M,N)$ is computed by the following sequence of steps

1. Make a resolution of $N$ by free left $A$-modules, $C_*$.
2. Tensor $C_*$ with $M$: $M \otimes_A C_*$.
3. Take homology:

$$H_i(M \otimes_A C_*) = \mathrm{Tor}_i^A(M,N).$$

From the previous section, $B_*(A)$ gives a free resolution of $k$. So for a right $A$-module $M$,

$$\mathrm{Tor}_i^A(M,k) = H_i(M \otimes_A B_*(A)).$$

**Definition 2.4.** *The* bar complex *for $A$ with coefficients in $k$ is*

$$\overline{B}_*(A) = k \otimes_A B_*(A).$$

*The* cobar complex *for $A$ with coefficients in $k$ is*

$$\overline{B}^*(A) = Hom_A(B_*A, k).$$

Since the bar resolution was exact, after tensoring we at least still have a complex, though it may or may not be exact. Furthermore,

$$\overline{B}_n(A) = k \otimes_A A^{\otimes(n+1)} = k \otimes_A A \otimes_k A^{\otimes n} \cong A^{\otimes n}.$$

Confusingly then, the bar complex (as groups) is a shift of the bar resolution. But the differential is not shifted, and the bar complex is a complex of $k$-vector spaces only, but not a complex of left $A$-modules.

With this definition,

$$H_i(\overline{B}_*(A)) = \mathrm{Tor}_i^A(k,k) =: H_i(A;k).$$

2.2. **The bar resolution and Ext.** Recall that for left $A$-modules, $M, N$, $\mathrm{Ext}_A(M,N)$ is computer by the following sequence of steps.

(1) Make a resolution of $M$ by free left $A$-modules $C_*$.
(2) Make a cochain complex by mapping into $N$:

$$C^* = \mathrm{Hom}_A(C_*, N)$$

(3) Take cohomology:

$$H^i(C^*) = \mathrm{Ext}^i_A(M,N).$$

From the previous section, $B_*(A)$ gives a free resolution of $k$. and

$$H^i(\overline{B}^* A) = \mathrm{Ext}^i_A(k,k) =: H^i(A;k).$$

## 3. The filtered bar and cobar complexes

Suppose $A$ has an algebra filtration as in Definition 1.3. This gives a filtration of complexes as follows.

$$F^p B_n(A) = \sum_{p_0+\cdots+p_n=p} F^{p_0}A \otimes_k F^{p_1}A \otimes_k \cdots \otimes_k F^{p_n}A \subseteq B_n(A).$$

There is something to check. We need to know that $d_n : F^p B_n(A) \to B_{n-1}A$. Using the multiplicative property of the filtration, it is clear that

$$a_0|\cdots|a_n \in F^p B_n(A) \Rightarrow a_0|\cdots|a_i a_{i+1}|\cdots|a_n \in F^p B_{n-1}(A).$$

We also need to understand what happens with the last term of the differential on the bar resolution. In the case of an increasing filtration, $\epsilon(a_n) \in k \subseteq F_0 A$. So that term of the differential may live in a smaller filtration, but certainly stays in $F^p$.

In the case of a decreasing filtration, if $a_n$ is in $F^p A$ for $p > 1$, $\epsilon(a_n) = 0$, so that last term of the differential is 0 (which is in all filtrations). Otherwise, $a_n \in F^0 A$ and no higher filtration, and $\epsilon(a_n)$ is also in $F^0$.

**Lemma 3.1.** *Suppose $A$ is a filtered algebra. The filtration on $A$ induces a filtration on $\overline{B}_*(A)$.*

*Proof.* Recall

$$\overline{B}_n(A) = k \otimes_A B_n(A) \cong A^{\otimes n}.$$

It will be helpful to have an explicit formula for the differential on this complex. Of course this is derived from the differential on $B_*A$.

$$\overline{d}_n[a_1|\cdots|a_n] = \epsilon(a_1)[a_2|\cdots|a_n] + \sum_{i=1}^{n-1}(-1)^i[a_1|\cdots|a_i a_{i+1}|\cdots|a_n] + (-1)^n[a_1|\cdots|a_{n-1}]\epsilon(a_n)$$

To understand the edge case ($n = 0, 1$) observer that $\overline{B}_0 A = k$ and $\overline{d}_1 = 0$.

Now that we have an explicit formula, we see that the same argument we used to see that the filtration on $A$ induced one on $B_*A$ applies in this situation. $\square$

**Lemma 3.2.** *Suppose $A$ is a filtered algebra. The filtration on $A$ induces a filtration on $\overline{B}^*(A)$. If the filtration on $A$ was increasing, the one on $\overline{B}^*(A)$ is decreasing, and vice versa.*

*Proof.* We note that

$$\overline{B}^n(A) = \mathrm{Hom}_A(B_nA, k) \cong Hom_k(\overline{B}_nA, k)$$

(in fact the second isomorphims hold for any left $A$-module $M$ in place of $k$). So, given an increasing filtration on $A$ we define

$$F^pB^nA = \{\gamma \in B^nA : \gamma|_{F^{p-1}B_nA} = 0\}.$$

If $A$ had had a decreasing filtration, we would have a similar definition, but with $\gamma|_{F^{p+1}} = 0$. □

## 4. Spectral Sequence

4.1. **The spectral sequences.** With the preliminaries out of the way, we now have spectral sequences computing $H_*(A; k)$ and $H^*(A; k)$. To identify the $E^1$ (or $E_1$) terms we have the following lemma. Recall that $E^0$ is our shorthand for the associated graded object consisting of the direct sum of the relative filtration quotients.

**Lemma 4.1.** *Let $A$ be a unital, associative, augmented, filtered algebra. Then*

$$E^0B_*A = B_*(E^0A)$$
$$E^0\overline{B}_*A = \overline{B}_*(E^0A)$$
$$E^0\overline{B}^*A = \overline{B}^*(E^0A).$$

Note that each of the complexes above are *bigraded*. We indicate the grading by $(s, t)$. The first index $s$ corresponds to tensor length, so that for example

$$B_{s,*}E^0A = (E^0A)^{\otimes n+1}.$$

This is called the "homological" (or when appropriate "cohomological") degree.

The second index, $t$, is the grading induced by the filtration, so that $B_{*,t}E^0A$ is the chain complex made by considering filtration $t$ at all levels module the next smaller filtration ($t-1$ or $t+1$ depending on whether this is an increasing or decreasing filtration). This is called the "internal" (or sometimes "topological") degree.

**Proposition 4.2.** *Let $A$ be a unital, associative, augmented algebra, with an increasing filtration. Then there is a homology spectral sequence with*

$$E^1_{p,q} = H_{p+q}(\overline{B}_{*,p}(E^0A)) = Tor^{E^0A}_{p+q,p}(k, k) = H_{p+q,p}(A; k).$$
$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$$
$$E^\infty_{p,q} = F^p\,Tor^A_{p+q}(k, k)/F^{p-1}\,Tor^A_{p+q}(k, k)$$

*There is a cohomology spectral sequence with*

$$E_1^{p,q} = H^{p+q}(\overline{B}^{*,p}(E^0A)) = Ext^{p+q,p}_{E^0A}(k, k) = H^{p+q,p}(A; k).$$
$$d_r^{p,q} : E_r^{p,q} \to E_n^{p+r,q-r+1}$$
$$E_\infty^{p,q} = F^p\,Ext^{p+q}_A(k, k)/F^{p+1}\,Ext^{p+q}_A(k, k)$$

If instead we have a decreasing filtration on $A$, we write $F^{-p}A$ for $F^p$ and we still get homology and cohomology spectral sequences respectively, with the same data.

For an increasing filtration with $F^{-1}A = 0$, our $E^1$ (and thus all $E^r$) term is 0 for $p < 0$ and for $p + q < 0$. Thus this spectral sequence is confined to the first quadrant plus the upper triangle of the fourth quadrant.

If in addition, $F^0A = k$, the spectral sequence is 0 when $q < 0$, so it is actually a first quadrant spectral sequence.

### 4.2. **Multiplication.**

**Proposition 4.3.** *The cobar complex, $\overline{B}^*(A)$ is an algebra. The differential is a derivation under this algebra structure.*

*Proof.* Let

$$c \in \overline{B}^n(A) = \mathrm{Hom}_k(\overline{B}_n(A), k), d \in \overline{B}^m(A).$$

Then $c \cdot d \in \overline{B}^{n+m}(A)$ is defined by

$$(c \cdot d)([a_1|\cdots|a_{n+m}]) = c([a_1|\cdots|a_n]) \cdot d([a_{n+1}|\cdots|a_{n+m}]).$$

This gives an associative multiplication, where $1 \in k = \overline{B}^0(A)$ is the unit.

We want to calculate the differential.

$$\begin{aligned}
\partial(c \cdot d)([a_1|\cdots|a_{n+m+1}]) &= (c \cdot d)(\delta[a_1|\cdots|a_{n+m+1}]) \\
&= (c \cdot d)(\epsilon(a_1)[a_2|\cdots|a_{n+m+1}] + \sum_{i=1}^{n+m}(-1)^i[a_1|\cdots|a_ia_{i+1}|\cdots|a_{n+m+1}] \\
&+ (-1)^{n+m+1}[a_1|\cdots|a_{n+m}]\epsilon(a_{n+m+1})) \\
&= c(\epsilon(a_1)[a_2|\cdots|a_{n+1}] + \sum_{i=1}^{n}(-1)^i[a_1|\cdots|a_ia_{i+1}|\cdots|a_{n+1}]) \cdot d([a_{n+2}|\cdots|a_{n+m+1}]) \\
&+ c([a_1|\cdots|a_n]) \cdot d(\sum_{i=n+1}^{n+m}(-1)^i[a_{n+1}|\cdots|a_ia_{i+1}|\cdots|a_{n+m+1}] \\
&+ (-1)^{n+m+1}[a_{n+1}|\cdots|a_{n+m}]\epsilon(a_{n+m+1}))
\end{aligned}$$

Note that

$$\begin{aligned}
(1)\quad c([a_1|\cdots|a_n]\epsilon(a_{n+1})) \cdot d([a_{n+2}|\cdots|a_{n+m+1}]) \\
&= c([a_1|\cdots|a_n]) \cdot \epsilon(a_{n+1}) \cdot d([a_{n+2}|\cdots|a_{n+m+1}]) \\
&= c([a_1|\cdots|a_n]) \cdot \cdot d(\epsilon(a_{n+1})[a_{n+2}|\cdots|a_{n+m+1}])
\end{aligned}$$

So we can add 0 to the expression for $\partial(c \cdot d)$ by adding the first expression from (1) (with the appropriate sign) and subtracting the last expression from (1) (with the same appropriate sign).

This gives precisely

$$\begin{aligned}
\partial(c \cdot d)([a_1|\cdots|a_{n+m+1}]) &= (\partial c)([a_1|\cdots|a_{n+1}]) \cdot d([a_{n+2}|\cdots|a_{n+m+1}) \\
&+ (-1)^n c([a_1|\cdots|a_n]) \cdot (\partial d)([a_{n+1}|\cdots|a_{n+m+1}]) \\
&= [(\partial c) \cdot d + (-1)^n c \cdot (\partial d)]([a_1|\cdots|a_{n+m+1}]).
\end{aligned}$$

In other words,

$$\partial(c \cdot d) = \partial c \cdot d + (-1)^n c \cdot \partial d.$$

□

**Remark**: The multiplication on $\overline{B}^*(A)$ induces the Yoneda multiplication on $\mathrm{Ext}^*_A(k,k)$.

**Remark**: If there is enough finiteness so that

$$\overline{B}^n(A) = (A^*)^{\otimes n}$$

(the tensor product of the vector space dual of $A$) then this product is the concatenation product.

**Proposition 4.4.** *Let $A$ be as in Proposition 4.2. Then the spectral sequence from Proposition 4.2 associated to the filtration of $\overline{B}^*(A)$ is a spectral sequence of algebras.*

*The multiplication on all $E^r$ terms is induced by the multiplication in $\overline{B}^*(A)$ so that the multiplication on the $E_0$ and $E_1$ terms is induced by is induced by the multiplication on $\overline{B}^*(E^0A)$.*

## 5. The universal enveloping algebra example.

Let $A = U(L)$ with the Lie filtration as in Section 1, and notation (e.g. basis for $L$) as in Section 1. So $L$ has basis $\{x_1, \ldots, x_n\}$, and

$$E^0U(L) = \mathrm{S}(L) = k[\overline{x}_1, \ldots, \overline{x}_n].$$

The overline on $\overline{x}_i$ isn't really necessary as this is an element of

$$F^1U(L)/F^0U(L) = (k \oplus L)/k = L$$

but it will be convenient to remind us where we are working.

In the cohomology spectral sequence from Proposition 4.2 for $A$,

$$E_1^{p,q} = \mathrm{Ext}^{p+q,p}_{S(L)}(k,k) = E(c_1, c_2, \ldots, c_n)$$

where each $c_i$ is $\mathrm{Ext}^{1,1}$ and can be defined as an element of $\mathrm{Hom}_k(\overline{B}_{1,1}(E^0A), k)$ by

$$c_i(\overline{x}_j) = \delta_{ij}.$$

Notice that this $E_1$-term is concentrated in degree $q = 0$, so although there can be a non-zero $d_1$, there can't be any higher differentials. Also observe that since $\mathrm{Ext}_{S(L)}(k,k)$ is dual to $\mathrm{Tor}^{S(L)}(k,k)$, we can make the same claim for the homology spectral sequence.

We will calculate the differentials in the cohomology spectral sequence via the following steps.

(1) Describe the $E^0$-term, $\overline{B}_*(S(L))$.
(2) Give a basis for $\mathrm{Tor}^{S(L)}_{1,1}(k,k)$ and $\mathrm{Tor}^{S(L)}_{2,2}(k,k)$.
(3) Use those bases to calculate $d^1_{1,0} : E^1_{1,0} \to E^1_{2,0}$.
(4) Dualize to get $d_1^{2,0} : E_1^{2,0} \to E_1^{1,0}$.
(5) Use the fact that $d_1$ is a derivation to deduce $d_1$ on the rest of $E_1^{p,0}$.

Step 1: We indicate the first few stages of the bar complex. The top row gives the homological degree.

$$
\begin{array}{ccccc}
0 & & 1 & & 2 \\
k & \longleftarrow & S(L) & \longleftarrow & S(L) \otimes_k S(L) \longleftarrow \\
0 & \underset{\epsilon(x)-\epsilon(x)}{\longleftarrow} & [x] & & \\
 & & \epsilon(x)[y] - [xy] + [x]\epsilon(y) & \longleftarrow & [x|y]
\end{array}
$$

Step 2: The equivalence classes in $H_1$ of the set of cycles

$$\{[\overline{x}_1], \ldots, \overline{x}_n\} \text{ is a basis for } \mathrm{Tor}_{1,1}.$$

The equivalence classes in $H_2$ of the set of cycles

$$\{[\overline{x}_i|\overline{x}_j] - [\overline{x}_j|\overline{x}_i]\}_{i<j} \text{ is a basis for } \mathrm{Tor}_{2,2}.$$

Step 3: We take a basis element for $\mathrm{Tor}_{2,2}$, take a representing cycle $[\overline{x}_i|\overline{x}_j] - [\overline{x}_j|\overline{x}_i]$ in $\overline{B}_2(S(L))$, and lift to an element of $\overline{B}_2(U(L))$, namely $[x_i|x_j] - [x_j|x_i]$. Note that this is in $F^2\overline{B}_*(A)$. Then we apply $d_2$ (the differential in $\overline{B}_*(A)$).

$$
\begin{aligned}
d_2([x_i|x_j] - [x_j|x_i]) &= \epsilon(x_i)[x_j] - \epsilon(x_j)[x_i] - ([x_i x_j] - [x_j x_i]) + [x_i]\epsilon(x_j) - [x_j]\epsilon(x_i) \\
&= 0 - [x_i x_j - x_j x_i] + 0 = [-[x_i, x_j]] = [-\lambda^k_{i,j} x_k] = -\lambda^k_{i,j}[x_k].
\end{aligned}
$$

We use the Einstein summation convention, so that in the expressions with a subscript $k$ and a superscript $k$, we sum over $k$.

In the expression with nested brackets, the outer bracket is notation from the bar complex. The inner bracket is the Lie bracket. The $\lambda^k_{i,j}$ are the structure constants in $L$, that is the Lie bracket

$$[x_i, x_j] = \lambda^k_{i,i} x_k$$

Step 4: We want to calcuate $d_1(c_k)$. We write

$$X_{i,j} = [\overline{x}_i|\overline{x}_j] - [\overline{x}_j|\overline{x}_i].$$

$$d_1(c_l)(X_{i,j}) = c_l(d^1(X_{i,j})) = c_l(-\lambda^k_{i,j} x_k) = -\lambda^l_{i,j}$$

The collection $\{c_i c_j\}_{i<j}$ forms a basis for $\mathrm{Ext}^{2,2}_{S(L)}$ dual to the given basis for $\mathrm{Tor}^{S(L)}_{2,2}$. We've just calculated

$$d_1(c_l) = \sum_{i<j} -\lambda^l_{i,j} c_i c_j = \sum_{i<j} \lambda^l_{i,j} c_j c_i. \tag{2}$$

Step 5: We just remark that we can extend $d_1$ to the rest of the $E_1$-term since we know it is a derivation and since the $c_l$ generate.

Finally we remark two things. The spectral sequence collapses at $E_2$ since $E_1^{p,q} = 0$ if $q \neq 0$. Also, since the $E_\infty$ term is 0 for $q \neq 0$, we get

$$\mathrm{Ext}^p_A(k, k) = E^{p,0}_\infty = E^{p,0}_2.$$

In other words $\mathrm{Ext}^*_A(k, k)$ is calculated from the homology of the complex $E(c_1, \ldots, c_n)$ with the given differential from equation (2).