The spectral sequence associated to an inclusion of a subspace.

Let $A \subseteq X$ be an inclusion. Let $C_*(X)$ be the chains on X. Define a filtration by

$$F^{-1}C_*(X) = 0, F^0C_*(X) = C_*(A), F^1C_*(X) = C_*(X) \tag{1}$$

It follows that $F^i = 0$ for $i < 0$ as well, and that $F^i = F^1$ for $i > 1$.

We get a spectral sequence associated to this filtration.

$$E^1_{p,q} = \begin{cases} 0 & p < 0, p > 1 \\ H_q(A) & p = 0 \\ H_{q+1}(X, A) & p = 1 \end{cases}$$

$d^1_{p,q} = 0$ except for $p = 1$ and

$$d^1_{1,q} : H_{q+1}(X, A) \to H_q(A)$$

is the connecting homomorphism, δ_{q+1} in the long exact sequence of the pair.

Then

$$E^2_{p,q} = \begin{cases} 0 & p < 0, p > 1 \\ \operatorname{cok}(\delta_{q+1}) & p = 0 \\ \ker(\delta_{q+1}) & p = 1 \end{cases}$$

$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is neccessarily identically 0 for $r > 1$, so that $E^2_{p,q} = E^3_{p,q} = \cdots = E^\infty_{p,q}$.

By consulting the LES of the pair, we deduce the following convergence result.

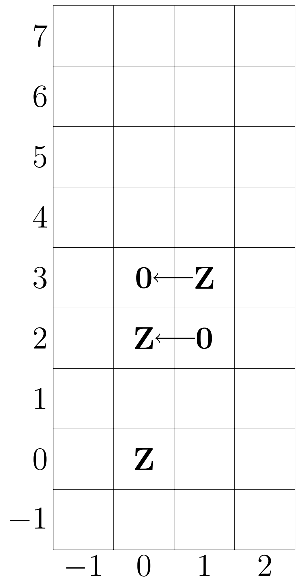
Theorem: For each q there is a short exact sequence

$$0 \to E^\infty_{0,q} \to H_q(X) \to E^\infty_{1,q-1} \to 0.$$

Example 1. Below is the E^1 page of the LES spectral sequence for the pair $S^2 \to \mathbf{C}P^2$. All empty squares represent the zero group, and I've included some $d^1_{p,q}$ so you can see why d^1 is forced to be identically z. The picture is based on the calculation

$$E^1_{p,q} = \begin{cases} 0 & p < 0, p > 1 \\ H_q(S^2) & p = 0 \\ H_{q+1}(\mathbf{C}P^2, S^2) & p = 1 \end{cases}$$

Note that the 0 groups indicated are superfluous since every empty square contains the 0 group.



E^1-page

We can deduce $H_*(\mathbf{C}P^2)$ by using the convergence statement

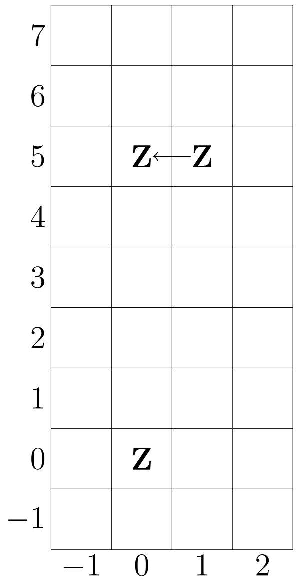
$$0 \to E^\infty_{0,q} \to H_q(\mathbf{C}P^2) \to E^\infty_{1,q-1} \to 0$$

is a short exact sequence, and using that $E^\infty_{p,q} = E^1_{p,q}$ in this example since all differentials are 0.

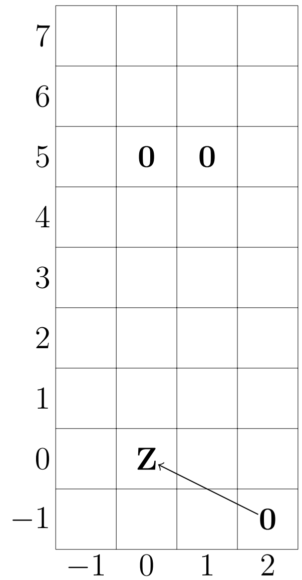
Example 2. Below is the E^1 page of the LES spectral sequence for the pair $S^5 \to D^6$, and the E^2 page. All empty square represent the zero group, so the picture is based on the calculation

$$E^1_{p,q} = \begin{cases} 0 & p < 0, p > 1 \\ H_q(S^5) & p = 0 \\ H_{q+1}(D^6, S^5) & p = 1 \end{cases}$$

and the differential d^1. In the E^2-page I include a sample d^2 for the purpose of illustrating that it has to be zero in this spectral sequence. d^2 connects columns that are two apart, therefore at least one of them has to be the zero column even in a more complicated pair of spaces.



E^1-page



E^2-page

If we know $H_*(S^5)$ and $H_*(D^6)$ we can deduce $H_*(D^6, S^5)$ by using

- $E^2_{p,q} = E^\infty_{p,q} = 0$ except that $E^2_{0,0} = \mathbf{Z}$.
- All columns other than the 0th and 1st are 0.
- Therefore $d^1_{1,q}$ is an isomorphism from the 1st column to the 0th column except when $q = 0$.

The spectral sequence associated to a filtered chain complex.

Let C_* be a chain complex. Assume that C_* has an increasing filtration

$$\cdots \subseteq F^{p-1}C_* \subseteq F^pC_* \subseteq F^{p+1}C_* \subseteq \cdots$$

We write F^p as shorthand for F^pC_* where possible.
We define a spectral sequence $\{E^r_{p,q}, d^r_{p,q}\}$ where $p, q \in \mathbf{Z}$, $r \in \mathbf{Z}_{\geq 0}$ by

$$E^0_{p,q} = F^pC_{p+q}/F^{p-1}C_{p+q}$$

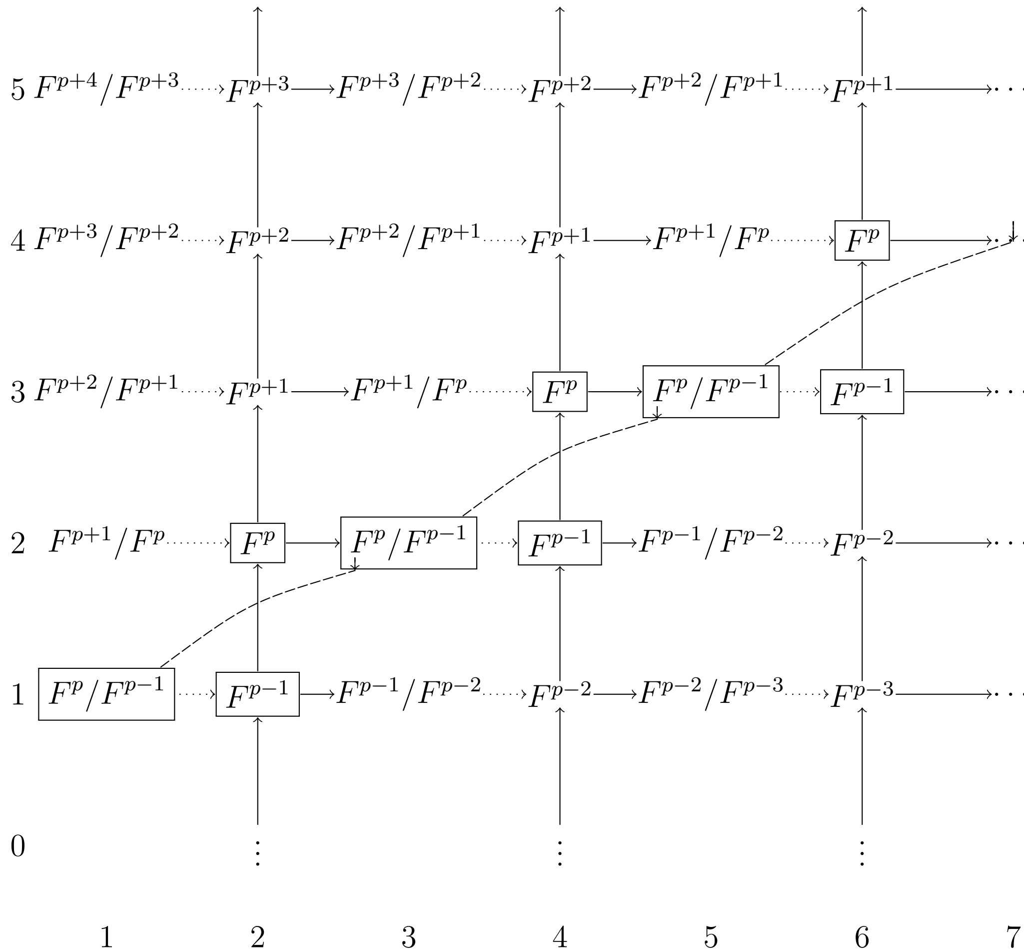
with $d^0_{p,q} : E^0_{p,q} \to E^0_{p,q-1}$ induced by the differential of C_*. Then

$$E^1_{p,q} = H_{p+q}(F^pC_*/F^{p-1}C_*)$$

with $d^1_{p,q} : E^1_{p,q} \to E^1_{p-1,q}$ the connecting homomorphism in the long exact sequence associated to

$$0 \to F^{p-1}/F^{p-2} \to F^p/F^{p-2} \to F^p/F^{p-1} \to 0.$$

This determines $E^2_{p,q}$ of course, but then I need to describe $d^2_{p,q}$, $d^3_{p,q}$, etc. There is a somewhat analogous way to proceed, but it is actually better to develop the picture known as the "unraveled exact couple." (We will delay, possibly forever, defining "exact couple.")



We will use this diagram to understand the E^r-terms of the spectral sequence.

<u>The E^0-page and d^0:</u>

The E^0 page of the spectral sequence is rarely explicitly specified in computational practice, but it is

$$E^0_{p,q} = F^pC_{p+q}/F^{p-1}C_{p+q}.$$

To keep notation precise, we suppressed the C_*. Everyplace in the diagram that F^k appears is really shorthand for F^kC_n (for some value of n). The vertical arrows are inclusions of subcomplexes, and thus preserve n. The solid horizontal arrows are quotient maps of chain complexes, so also preserve n. [The dotted horizonal arrows represent connecting homomophisms - to be discussed when discussing the E^1-page, which decrease n by 1.]

This diagram potentially occupies the entire plane, so there is a spot corresponding to F^sC_t for all possible s and t (moving up increases s and preserves t, moving right past a dotted arrow decreases t). To help keep score, if the bottom left location with a box around it is really the group

$$F^pC_n/F^{p-1}C_n = E^0_{p,n-p} \tag{2}$$

then the top right location with a box around it is the group

$$F^pC_{n-3}.$$

The spots in the diagram of the form F^pC_k are not part of the E^0 page, but those parts of the diagram are needed for calculation the d^r differentials when $r > 0$. [In the language of exact couples, those spots form the D^0 of the exact couple.]

Since $\{F^pC_*\}_{p\in\mathbf{Z}}$ is a filtration of the chain complex C_*, the differential on C_* induces differentials on each F^pC_* and thus also on each $F^pC_*/F^{p-1}C_*$. This gives the differential d^0 on the $E^0_{*,*}$ page of the spectral sequence.

The dashed curved arrows indicate the differentials in the chain complex F^p/F^{p-1}. Note that they go right two units and up one unit in the diagram. Of course all the other terms in the diagram are also parts of chain complexes, and the differentials also go right two terms and up one term.

In this diagram, we have short exact sequences of chain complexes (one for each p):

$$0 \to F^{p-1}C_* \to F^pC_* \to F^p/F^{p-1} \to 0. \tag{3}$$

The E^1-page and d^1:

To get the E^1-page, recall that each group in the diagram is part of a chain complex, where the differential moves right two places and up one place. Form a new diagram, associated to the E^1-page by replacing each group in the diagram with its homology. Here we reproduce the first five columns of the previous diagram.

We don't really need a new diagram, we could just mentally replace each group in the initial diagram with its homology. So following the convention in (2) the lower left spot that has a box around it, we have

$$H_n(F^pC_*/F^{p-1}C_*) = E^1_{p,n-p}.$$

The top right location from the first diagram with a box around now contains the group

$$H_{n-3}(F^pC_*)$$

but we haven't duplicated the rightmost columns in the new diagram, so the rightmost spot in the new diagram with a box around it is

$$H_{n-2}(F^p/F^{p-1}) = E^1_{p,n-2-p}.$$

The short exact sequence (3) induces a long exact sequence

$$\cdots \to H_n(F^p/F^{p-1}) \to H_{n-1}(F^{p-1}) \to H_{n-1}(F^p) \to \cdots \tag{4}$$

The dotted arrows in the diagram correspond to the connecting homomorphism in the long exact sequence, so that the boxed entries in the diagram above form the long exact sequence (4).

Of course this diagram has many long exact sequences. For example, if we were to draw circles around each group that is directly above a group with a box around it, the circled groups would give another long exact sequence

$$\cdots \to H_n(F^{p+1}/F^p) \to H_{n-1}(F^p) \to H_{n-1}(F^{p+1}) \to \cdots$$

The two exact sequences overlap at $H_*(F^p)$. This diagram is best thought of as a collection of interlocking long exact sequences

Two sample d^1 maps are indicated by the curved dashed arrows. Of course there are more d^1 maps in the diagram in the rows above and below the row in which we've drawn the sample d^1 maps.

$$d^1 : E^1_{p,q} \to E^1_{p-1,q} \qquad [\text{that is } H_{p+q}(F^p/F^{p-1}) \to H_{p+q-1}(F^{p-1}/F^{p-2})]$$

is the composite of the two obvious horizontal maps. The first map is the dotted map, which reduces the homology dimension by 1.

It is obvious that $d^1_{p-1,q} \circ d^1_{p,a} = 0$ because the composite contains two consecutive maps in one of the interlocking long exact sequences.

<u>The E^2-page and d^2:</u>

Of course

$$E^2_{p,q} = H_{p,q}(E^1_{*,*}, d^1_{*,*})$$

so $E^2_{p,q}$ is a subquotient (a subgroup of a quotient group or equivalently a quotient group of a subgroup) of $E^1_{p,q} = H_{p+q}(F^pC_*/F^{p-1}C_*)$. (In fact it is a subqotient of $F^pC_{p+q}/F^{p-1}C_{p+q}$.)

<u>We describe heuristically how to calculate d^2.</u> Consider an element in the middle of our diagram that "lives to" the E^2 page. That is,

$$\overline{x} \in E^2_{p+1,n-p-2} \text{ so that a representative is } x \in H_{n-1}(F^{p+1}/F^p).$$

$d^1(x) = 0$, so if you push x two spots to the right, you get 0. It follows that if you push x *one* spot to the right (call this element $\delta(x)$) then you get an element in $H_{n-2}F^p$ which is in the image of $H_{n-2}F^{p-1}$. Denote a preimage of $\delta(x)$ by $y \in H_{n-2}F^{p-1}$.

Push y one spot to the right. Denote this $j(y) \in H_{n-2}(F^{p-1}/F^{p-2})$. By the long exact seqeunce at that spot, $d^1(j(y)) = 0$, so $j(y)$ yields an element in the homology with respect to d^1, and thus in $E^2_{p-1,q-1}$.

Reread the above two paragraphs, putting $x, y, \delta(x), j(y)$ into the diagram until it makes perfect sense to you. There is nothing in this write-up as critical to understand!

In terms of the maps in our diagram, we denote all vertical maps by i, all horizontal maps out of some H_*F_p for some p by j, and the dotted arrows by δ. Then

$$d^2(\overline{x}) = \overline{(j \circ i^{-1} \circ \delta)(x)}$$

In other words, d^2 is calculated by moving one place to the right, one place *down*, and then another place to the right.

Why is this a heuristic description? Only because there is work to do to check that d^2 is well-defined (we chose a representative for $\overline{x}$ and we chose a preimage of $\delta(x)$).

Note that as in the definition of d^1, it is clear that $(d^2)^2 = 0$ as long as we believe d^2 is defined, because $(d^2)^2 = 0$ involves a $\delta \circ j$, which is zero since it is two consecutive maps in an exact sequence.

Lemma 0.1. *$d^2_{p,q}$ is well-defined as an element of $E^2_{p-2,q+1}$; that is it is independent of the various choices made above.*

The E^r-page and d^r:

Recall

$$E^r_{p,q} = H_{p,q}(E^{r-1}_{*,*}, d^{r-11}_{*,*})$$

so $E^r_{p,q}$ can be viewed as a subquotient of $E^{r-1}_{p,q}$ and then by transitivity as a subquotient of $E^1_{p,q} = H_{p+q}(F^pC_*/F^{p-1}C_*)$. (Or even $F^pC_{p+q}/F^{p-1}C_{p+q}$.)

We describe heuristically how to calculate d^r. Consider an element x in the middle of our diagram that "lives to" the E^r page. That is, first it "lives to" the E^{r-1} page and then $d^{r-1}(\overline{x}) = 0$.

By induction,

$$d^{r-1}(\overline{x}) = \overline{(j \circ i^{2-r} \circ \delta)(x)}.$$

Thus

$$j[(i^{2-r} \circ \delta)(x)] = 0 \text{ in the } E^{r-1} \text{ term.}$$

One can show that with an appropriate choice of reprentative x, and appropriate life $[(i^{2-r} \circ \delta)(x)]$ we get

$$j[(i^{2-r} \circ \delta)(x)] = 0 \text{ in the } E^1 \text{ term.}$$

So by the long exact sequence in the E^1-term, $i^{1-r}(\delta(x))$ can be chosen. Then

$$d^r(\overline{x}) = \overline{j \circ i^{1-r} \circ \delta(x)} \tag{5}$$

Lemma 0.2. *$d^r_{p,q}$ is well-defined as an element of $E^2_{p-r,q+r-1}$; that is it is independent of the various choices made above.*

Rather than prove these lemmas as stated, I'd like to reformulate the spectral sequence in a way that makes it easy to prove things.

The unraveled exact couple: cycles and boundaries.

Let F^pC_* ($p \in \mathbf{Z}$) be a filtered chain complex. Recall this means that

$$d|_{F^pC_{p+q}} : F^pC_{p+1} \to F^pC_{p+q-1}.$$

Recall also that

$$F^pC_{p+q-1} \supseteq F^{p-1}C_{p+q-1} \subseteq F^{p-2}C_{p+q-2} \supseteq \cdots$$

Definition 0.3. *Define the set of r-cycles by*

$$Z^r_{p,q} = d^{-1}F^{p-r-1}C_{p+q-1} + F^{p-1}C_{p+q} \subseteq F^pC_{p+q}$$

Definition 0.4. *Define the set of r-boundaries by*

$$B^r_{p,q} = [d(F^{p+r}C_{p+q+1}) \cap F^pC_{p+q}] + F^{p-1}C_{p+q}.$$

These definitions are the same as the ones I gave in class though phrased differently. The $Z^r_{p,q}$ are nested so that larger rs give rise to smaller groups.

The $B^r_{p,q}$ are nested so that larger r give larger groups.

Lemma 0.5. *Let r, s be integers greater than 0. Then $B^r_{p,q} \subseteq Z^s_{p,q}$ with no other restrictions on r and s.*

The following definition makes explicit the idea in (5).

Definition 0.6.

$$d^{r+1}_{p,q} : Z^r_{p,q}/B^r_{p,q} \to Z^r_{p-r,q+r-1}/B^r_{p-r,q+r-1}$$

on $\overline{x}$ by choosing a representative x so that $dx \in F^{p-r}C_{p+q-1}$

Lemma 0.7. *Take $r \geq 1$.*

(1) *d^r is well defined.*
(2) *If we take $E^{r+1} = Z^r/B^r$, then $H_*(E^r, d^r) = E^{r+1}$.*
(3) *$E^1_{p,q} = H_{p+q}(F^pC_*/F^{p-1}C_*)$.*