# Spectral Sequence Notes: Finiteness of homotopy groups

## 1. Finite type and the Serre Spectral Sequence

**Definition 1.1.** *A space $X$ is* locally finite *if $H_i(X)$ is finitely generated for each $i$.*

This definition is motivated by the fact that if $X$ is locally finite, then a CW appoximation can be made that has finitely many cells in each dimension.

We're interested in understanding how locally finite is preserved in fibration. We'll start by considering the path-loop fibration for a simply connected base space $X$.

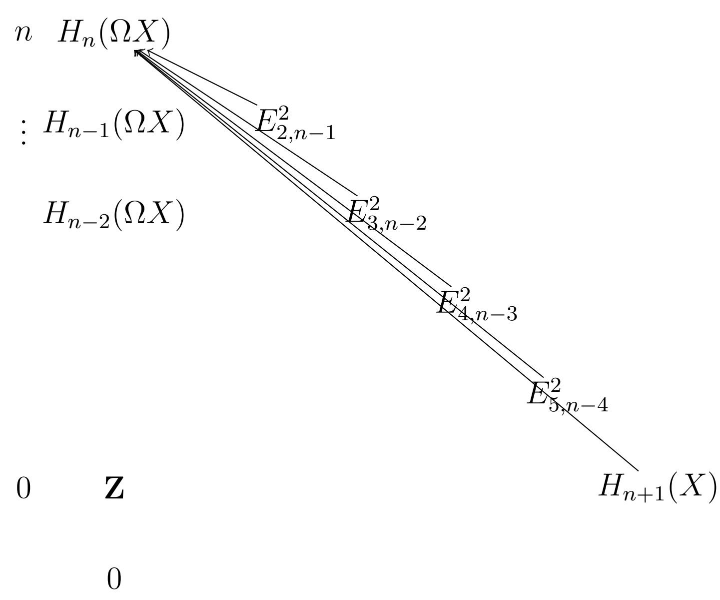
$$\begin{tikzcd} \Omega X \arrow[r] & PX \arrow[d, "p"] \\ & X \end{tikzcd} \tag{1}$$

Assume that $X$ is locally finite, and let $i \geq 2$ be the first dimension in which $X$ has non-zero reduced homology. Then $\Omega X$ is $i-2$ connected and

$$H_{i-1}(\Omega X) = \pi_{i-1}(\Omega X) = \pi_i(X) = H_i(X)$$

by using the Hurewicz theorem together with the shift in homotopy corresponding to $\Omega$. So $H_{i-1}(\Omega X)$ is finitely generated. We will do induction, and this serves as our base space. [You might be worried about the case $i - 1 =$ in which case Hurewicz won't necessarily give an isomorphism, but note that $\pi_{i-1}(\Omega X)$ is abelian, so that won't be an issue.]

Now suppose $H_k(\Omega X)$ is finitely generated for $k < n$ and consider the $E^2$ term of the Serre Spectral Sequence for our path loop fibration.




By induction (since tensor products of finitely generated groups are finitely generated) everything group in a spot below the line $q = n$ is finitely generated. It follows that $E^2_{2,n-2} E^3_{3,n-3}, \ldots, E^{n+1}_{n+1,0}$ are finitely generated since they are subquotients of finitely generated groups.

Now we note that the relevant $E^\infty$ groups are zero, so the differential $d^{n+1} : E^{n+1}_{n+1,0} \to E^{n+1}_{0,n}$ is an isomorphism. Thus $E^{n+1}_{0,n}$ is finitely generated. Next note there is a short exact sequence

$$0 \to \mathrm{im}(d^n_{n,1}) \to E^n_{0,n} \to E^{n+1}_{0,n} \to 0.$$

We just showed the right hand group is finitely generated, and since the source of $d^n_{n,1}$ is finitely generated so is its image. It follows that $E^n_{0,n}$ is finitely generated.

Similarly

$$0 \to \mathrm{im}(d^{n-1}_{n-1,2}) \to E^{n-1}_{0,n} \to E^n_{0,n} \to 0.$$

has finitely generated groups at the ends, and thus $E^{n-1}_{0,n}$ is finitely generated.

We iterated this argument for $d^{n-k}_{n-k,k+1}$, finishing with $d^2_{2,n-1}$ to show that

$$E^2_{0,n} = H_n(\Omega X)$$

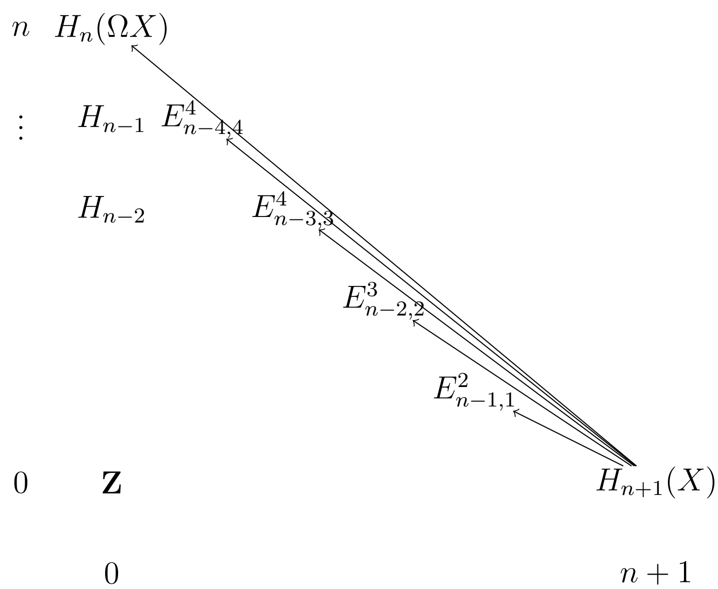
is finitely generated.

We can mimic this argument by using the path-loop fibration (1) but now assuming that $\Omega X$ is locally finite. If $i$ is the first dimension in which $\Omega X$ has non-zero reduced homology, then

$$H_{i+1}(X) = \pi_{i+1}(X) = \pi_i(\Omega X) = H_i(\Omega X).$$

So it follows that $H_{i+1}(X)$ is finitely generated (and lower dimensional reduced homology groups are 0).

Now suppose $H_k(X)$ is finitely generated for $k \leq n$, and consider the $E^2$-term of the Serre Spectral Sequence for the path-loop fibration (1). We'd like to show $H_{n+1}(X)$ is finitely generated.




Our argument is very similar. The relevant $E^\infty$ groups are 0. Since $E^\infty_{n+1,0} = E^{n+2}_{n+1,0}$, we have $d^{n+1}_{n+1,0}$ is an isomorphism. The target is finitely generated, so $E^{n+2}_{n+1,0}$ is finitely generated.

Now if we assume by induction that $E^k_{n+1,0}$ is finitely generated for $2 < k \leq n+2$, we examine the differential

$$E^{k-1}_{n+1,0} \to E^{k-1}_{n+2-k,k-2}$$

$E^k_{n+1,0}$ is the kernel of that map, and the image is finitely generated as it is a subgroup of a finitely generated group. So we have a short exact sequence

$$0 \to E^k_{n+1,0} \to E^{k-1}_{n+1,0} \to \operatorname{im}(d^{k-1}_{n+1,0}) \to 0.$$

Since the two groups on the end are finitely generated, so is the group in the middle.

We conclude that $E^2_{n+1,0} = H_{n+1}(X)$ is finitely generated. In other words, under the original hypotheses (including $X$ simply connected), $X$ is finite type if and only if $\Omega X$ is finitely generated.

These methods generalize.

**Proposition 1.2.** *Suppose we have a fibration*

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "p"] \\ & B \end{tikzcd} \tag{2}$$

*with $F$ path connected and $B$ simply connected. If two of the three spaces $F, E, B$ are finite type, so is the third.*

*Proof.*
(1) Suppose $B, E$ are finite type. We proceed as in the first case above when considering the path-loop fibration (1). The only difference between this case and that one is that the fact that $E$ is locally finite implies all $E^\infty_{p,q}$ are fintely generated. Thus the differential
$$d^{n+1}_{n+1,0} : E^{n+1}_{n+1,0} \to E^{n+1}_{0,n}$$
has finitely generated cokernel and finitely generated image. So $E^{n+1}_{0,n}$ is finitely generated. The rest of the proof proceeds exactly as in the case above where we are showing $X$ finite type imples $\Omega X$ finite type.

(2) Suppose $E, F$ are finite type. We proceed as in the second case above considering the path-loop fibration (1). The only difference bewteen this argument and that case is that since $E$ is locally finite, so are all $E^\infty_{p,q}$ are finitely generated. Thus the differential
$$d^{n+1}_{n+1,0} : E^{n+1}_{n+1,0} \to E^{n+1}_{0,n}$$
has a finitely generated kernel, establishing that $E^{n+2}_{n+1,0}$ is finitely generated. The rest of the proof proceeds as in the case where we show $\Omega X$ finite type implies $X$ finite type.

(3) Suppose $F, E$ finite type. Then each $E^2_{p,q}$ is finitely generated. So each $E^\infty_{p,q}$ is finitely generated. Since $H_n(E)$ is made from a finite series of extensions involving $E^\infty_{n-q,q}$, it follows that $H_n(E)$ is finitely generated.

□

This result can be generalized much further.

**Definition 1.3.** *A* Serre class *of abelian groups is a class of abelian groups $\mathcal{C}$ satisfying*

(1) *For any short exact sequence*
$$0 \to A \to B \to C \to 0$$
*the group $B \in \mathcal{C}$ if and only if $A$ and $C$ are both in $\mathcal{C}$.*

(2) *$A, B \in \mathcal{C}$ implies $A \otimes C \in \mathcal{C}$.*

Here are some useful example:

(1) finitely generated abelian group.
(2) finite abelian groups.
(3) torsion abelian groups.
(4) $p$-primary torsion abelian groups.
(5) $p$-local abelian groups.

Some non-examples include $k$-vector spaces.

**Proposition 1.4.** *Suppose we have a fibration*

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "p"] \\ & B \end{tikzcd} \tag{3}$$

*with $F$ path connected and $B$ simply connected. Let $\mathcal{C}$ be a Serre class of abelian groups. If two of the three spaces $F, E, B$ have all of their reduced homology groups in $\mathcal{C}$, so does the third.*

*Proof.* The proof is really the same as that of Proposition 1.2. □

One can actually generalize this further by considering homology in a PID $R$, and Serre classes of $R$-modules. One needs $R$ to be a PID so that the Künneth Theorem works in the expected way. This is useful, for example, when $R$ is a field to show that the "two out of three property" holds for homology with $R$ coefficients being finitely generated in each degree.

## 2. Qualtiative information about $H_*(K(A, n))$.

**Corollary 2.1.** *$K(\mathbf{Z}, n)$ is finite type.*

*Proof.* This is true for $n = 1, 2$ ($S^1$ and $\mathbf{C}P^\infty$). Assume it is true for $K(\mathbf{Z}, n)$. Then consider the path loop fibration

$$\begin{tikzcd} K(\mathbf{Z}, n) \arrow[r] & PK(\mathbf{Z}, n+1) \arrow[d, "p"] \\ & K(\mathbf{Z}, n+1) \end{tikzcd}$$

The fiber and the total space are finite type, so by Proposition 1.2 so is the base ($n \geq 1$, so $K(\mathbf{Z}, n+1)$ is simply connected). □

**Corollary 2.2.** *$K(\mathbf{Z}/(l), n)$ is finite type.*

*Proof.* For $n \geq 2$, we consider the fibration

$$\begin{tikzcd} F \arrow[r] & K(\mathbf{Z}, n) \arrow[d, "p"] \\ & K(\mathbf{Z}/(l), n) \end{tikzcd}$$

where the map $p$ is chosen so that on homotopy it induces the quotient map $\mathbf{Z} \to \mathbf{Z}/(l)$. It follows, by examination of the long exact sequence of the fibration that $F$ is a $K(\mathbf{Z}, n)$.

We apply Proposition 1.2 where the fiber and total space are known to be finite type by Corollary 2.1. We conclude $K(\mathbf{Z}/(l), n)$ is finite type for $n \geq 2$. This doesn't work for $n = 1$ since then the base space is not simply connected.

For $n = 1$ we consider the fibration

$$\begin{array}{ccc} K(\mathbf{Z}/(l),1) & \longrightarrow & PK(\mathbf{Z}/(l),2) \\ & & \big\downarrow p \\ & & K(\mathbf{Z}/(l),2) \end{array}$$

The base and the total space are finite type and the base is simply connected, so by Proposition 1.2, so is the fiber. □

We can do better than just finite type for $K(\mathbf{Z}/(l),n)$.

**Corollary 2.3.** *$H_i(K(\mathbf{Z}/(l),n))$ is a finite abelian group for each $i$.*

*Proof.* We first need to prove this for $K(\mathbf{Z}/(l),1)$, which is also the classifying space $B\mathbf{Z}/(l)$. One can explicitly calculate the homology of this space by viewing it as in infinite dimensional lens space. One gets

$$\tilde{H}_i(K(\mathbf{Z}/(l),1)) = \begin{cases} 0 & i \text{ even} \\ \mathbf{Z}/(l) & i \text{ odd} \end{cases}$$

THere is also a cheap trick one can do that generalizes to other finite groups (even if they aren't abelian). Consider the universal cover of $K(\mathbf{Z}/(l),1)$, call it $E$ and let $p : E \to K(\mathbf{Z}/(l),1)$ be the corresponding covering map. We'll think about the map

$$p_* : \tilde{H}_*(E) \to \tilde{H}_*(K(\mathbf{Z}/(l),1).$$

All homotopy groups (and thus all reduced homology groups) of $E$ are 0. So $p_*$ doesn't look very promising. But there is also a map the other way. called the *transfer*

$$\tau : \tilde{H}_*(K(\mathbf{Z}/(l),1) \to \tilde{H}_*(E).$$

$\tau$ is not induced by a map on spaces, but it is induced by a map on singular chains. If $\sigma : \Delta^p \to K(\mathbf{Z}/(l),1)$ is a singular simplex, then for each $e \in p^{-1}(\sigma(v_0))$ there is a unique lift of $\sigma$ to a map sending $v_0$ to $e$.

$\tau$ is defined on chains by sending $\sigma$ to the sum of these lifts. This gives a map on homology.

This construction works for any finite covering space of course. The interestesting thing about $\tau$ is that $p_* \circ \tau$ is multiplication by the order of the covering space.

In this case, $p_* \circ \tau$ is multiplication by $l$. But it factors through 0, so multiplication by $l$ is zero. We already know $H_i(K(\mathbf{Z}/(l),1))$ is finitely generated, but now we know each generator has order dividing $l$. So the maximum size of the group is the number of generators raised to the power $l$.

(This same argument tells you that if $G$ is any finite group, then multiplication by the order of $G$ is zero on the reduced homology groups.)

So we've dealt with the base case. Now assume we know that each reduced homology group of $K(\mathbf{Z}/(l),n-1)$ is finite. Consider the fibration

$$\begin{array}{ccc} K(\mathbf{Z}/(l),n-1) & \longrightarrow & PK(\mathbf{Z}/(l),n) \\ & & \big\downarrow p \\ & & K(\mathbf{Z}/(l),n) \end{array}$$

We know that the fiber and the total space have each reduced homology group finite. So applying Proposition 1.4 in the case that $\mathcal{C}$ is the class of finite abelian groups, we learn the same thing about the base of the fibration. $\square$

**Corollary 2.4.** (1) *If $A$ is finitely generated, then $K(A,n)$ is finite type.*
(2) *If $A$ is finite, then each $\tilde{H}_i(K(A,n))$ is finite.*

*Proof.* (1) $A$ is a product of cyclic groups. So $K(A,n)$ is a product of $K(C_i,n)$ for cyclic $C_i$. So its homology is determined by iterating the Künneth Theorem. The tensor and torsion product of finitely generated groups are also finitely generated.
(2) $A$ is a product of finite cyclic groups. So $K(A,n)$ is a product of $K(C_i,n)$ for finite cyclic $C_i$. The tensor and torsion product of finite groups are finite.

$\square$

**Proposition 2.5.** *For $n \geq 1$,*

$$H^*(K(\mathbf{Z},n),\mathbf{Q}) = \begin{cases} E(x_n) & n \text{ odd} \\ \mathbf{Q}[x_n] & n \text{ even} \end{cases}$$

*Proof.* Since $K(\mathbf{Z},1) \simeq S^1$ and $K(\mathbf{Z},2) \simeq \mathbf{C}P^\infty$, we have the result for $n = 1,2$. We assume the result for $n$ and use that to derive the result for $n+1$. We'll use the Serre Spectral Sequence associated to the path-loop fibration

$$\begin{array}{ccc} K(\mathbf{Z},n) & \longrightarrow & PK(\mathbf{Z},n+1) \\ & & \downarrow{\scriptstyle p} \\ & & K(\mathbf{Z},n+1) \end{array}$$

Case 1: $n$ odd. The $E_2$-page looks like

$$\begin{array}{cccccccc}
 & 0 & & & & & & 0 \\
q=n & \mathbf{Q} & 0 & \cdots & \cdots & \cdots & 0 & \mathbf{Q} \\
 & 0 & & & & & & 0 \\
 & \vdots & & & & & & \vdots \\
 & \vdots & & & & & & \vdots \\
 & 0 & & & & & & 0 \\
q=0 & \mathbf{Q} & 0 & \cdots & \cdots & \cdots & 0 & \mathbf{Q} \\
 & p=0 & & & & & & p=n+1
\end{array}$$

Only the 0th row and the $n$th row can have non-zero entries by the assumption about $K(\mathbf{Z}, n)$. The Hurewicz Theorem gives us the 0th row from $p = 0$ through $p = n + 1$. Since the homology (and thus the cohomology) groups are finitely generated and the base is simply connected,

$$E_2^{p,q} = H^p(K(\mathbf{Z}, n+1); \mathbf{Q}) \otimes H^q(K(\mathbf{Z}, n); \mathbf{Q}).$$

We write $x_n$ for the generator of $H^n(K(\mathbf{Z}, n); \mathbf{Q})$ and $x_{n+1}$ for the generator of $H^{n+1}(K(\mathbf{Z}, n+1); \mathbf{Q})$. Thus the four copies of $\mathbf{Q}$ indicated in the picture have generators $1 \otimes 1, x_{n+1} \otimes 1, 1 \otimes x_n, x_{n+1} \otimes x_n$.

Because of the vanishing of all but two rows, $E_2 = E_3 = \cdots = E_{n+1}$ and $E_{n+2} = E_\infty$. Because $E_\infty^{p,q} = 0$ unless $p = q = 0$, we have

$$d_{n+1}^{p,q} : E_{n+1}^{p,q} \xrightarrow{\cong} E_{n+1}^{p+n+1,q-n}$$

for $q = n$.

Then WLOG, choose $x_{n+1}$ so that $d_{n+1}(1 \otimes x_n) = (x_{n+1} \otimes 1)$. For $n + 1 < p < 2n + 2$, we get

$$H^p(K(\mathbf{Z}, n+1); \mathbf{Q}) \cong E_2^{p,0} \cong E_{n+1}^{p,0} \cong E_{n+1}^{p-n-1,n} = 0.$$

For $p = 2n + 2$, we still have $d_{n+1}^{n+1,n}$ is an isomorphism landing in $E_{n+1}^{2n+2,0} = H^{2n+2}(K(\mathbf{Z}, n+1); \mathbf{Q})$. But we also have (using that this is a spectral sequence of algebras)

$$d(x_n \otimes x_{n+1}) = d(x_n \otimes 1)(1 \otimes x_{n+1}) + (x_n \otimes 1)d(1 \otimes x_{n+1}) = 1 \otimes x_{n+1}^2.$$

So $H^{2n+2}(K(\mathbf{Z}, n+1); \mathbf{Q}) \cong \mathbf{Q}$ with generator $x_{n+1}^2$.

If one assumes inductively that $H^*(K(\mathbf{Z}, n+1); \mathbf{Q}) \cong \mathbf{Q}[x_{n+1}]$ through dimension $* = k(n + 1)$ then a similar argument tells us that $H^p(K(\mathbf{Z}, n + 1); \mathbf{Q}) = 0$ for $k(n+1) < p < (k+1)(n+1)$ and that $d_{n+1}^{k(n+1),n}$ is an isomorphism taking $x_{n+1}^k \otimes x_n$ to $x_{n+1}^{k+1}$. So

$$H^{(k+1)(n+1)}(K(\mathbf{Z}, n + 1); \mathbf{Q}) \cong \mathbf{Q} \text{ generated by } x_{n+1}^{k+1}$$

Then by induction, $H^*(K(\mathbf{Z}, n + 1); \mathbf{Q})$ is as claimed.

Case 1: $n$ even. We think about $E_2$-page using the same diagram above. But this is a less complete description since there is a non-zero row at each $q = kn$. We know $H^*(K(\mathbf{Z}, n + 1); \mathbf{Q})$ for $* \leq n + 1$ as before by the Hurewicz Theorem.

This tells us complete information about the columns $p = 0$ through $p = n + 1$.

We also know that at locations $(0, n)$ and $(n + 1, 0)$, we have both $E_2 = E_{n+1}$ as before, and $E_{n+2} = E_\infty$. So $d_{n+1}$ is an isomorphism between these locations, so as before choose a generator $x_{n+1} \in H^{n+1}(K(\mathbf{Z}, n + 1); \mathbf{Q})$ so that $d_{n+1}^{0,n}(1 \otimes x_n) = x_{n+1} \otimes 1$. With this notation, the non-zero groups in the $p = 0$ column are generated by $1 \otimes x_n^k$ and the non-zero groups in the $p = n + 1$ column are generated by $x_{n+1} \otimes x_n$.

Using that $d_{n+1}$ is a derivation on the $E_{n+1}$-page and that $d_{n+1}(x_{n+1} \otimes 1) = 0$, we calculate

$$d(x_{n+1} \otimes x_n^k) = k(x_{n+1} \otimes x_n^{k-1})$$

which is an isomorphism for $k \geq 1$.

It follows that $E_{n+2}^{p,q} = 0$ for $p \leq n + 1$ (except at $(0, 0)$). We now want to ask if $H^k(K(\mathbf{Z}, n + 1); \mathbf{Q})$ can be non-zero for $k > n + 1$. Suppose $k$ is the smallest interger above $n + 1$ for which $H^k(K(\mathbf{Z}, n + 1); \mathbf{Q})$ is not known to be zero. So in the $E_2$-term, the $p=0, p=n+1$ columns are non-zero, and the next column that might be non-zero is $p=k$.

Of course all differentials leaving the $(k,0)$ spot are 0, so each $E_r^{k,0}$ is a quotient of $H^k(K(\mathbf{Z}, n+1); \mathbf{Q})$. Also, any differentials into the $(k,0)$ spot must come from the $p=n+1$ or $p=0$ column. But we've already established that everything in the $p=n+1$ column is hit by a differential, so it cannot also support a differential. We've also established that nothing in the $p=0$ is in the kernel of $d_{n+1}$, so there is nothing there to support a $d_k$.

Thus all differentials into the $(k,0)$ spot are 0. So

$$0 = E_\infty^{k,0} = E_2^{k,0} = H^k(K(\mathbf{Z}, n+1); \mathbf{Q}).$$

Thus $H^*(K(\mathbf{Z}, n+1); \mathbf{Q})$ is as claimed. $\square$

**Remark:** We haven't determined the integral homology, but if we combine Proposition 2.5 with Corollary 2.1 (and the universal coefficient theorem when appropriate) we see that in dimensions with zero rational homology, we get finite homology and cohomology groups, and in dimensions with a copy of $\mathbf{Q}$ in the rational cohomology, we get homology and cohomology groups that are a direct sum of $\mathbf{Z}$ with some finite group.

## 3. Spheres

We will make repeated use of the following facts.

(1) Let $X$ be CW, $A$ an abelian group. Then $H^n(X, A) = [X, K(A, n)]$.
(2) If $X$ is $n-1$-connected and $A = \pi_n(X) = H_n(X)$ then

$$\mathrm{Hom}(A, A) = \mathrm{Hom}(H_n(X), A) \cong H^n(X; A) \cong [X, K(A, n)]$$

and the map $\iota_A : X \to K(A, n)$ on the right hand side corresponding to the identity of $A$ on the left hand side induces an isomorphism in $\pi_n$.
(3) Given any map $f : X \to Y$, one can replace $X$ with a homotopy equivalent space $X'$ and $f$ with a corresponding map $f'$ so that $f'$ is a fibration. The homotopy type of the fiber of $f'$ is independent of how one does this replacement.

These are the facts that allow one to imitate the universal covering space construction to eliminate higher homotopy groups in the way that the universal cover eliminates the fundamental group. Unfortunately, one doesn't get covering spaces doing this, just fibrations.

More precisely, let $X$ be an $n-1$-connected CW complex. Let $A = \pi_n(X)$. Use the facts above to make a map $X \to K(A, n)$ which is an isomorphism on $\pi_n$, and replace $X$ with a homotopy equivalent space to make this map a fibration. We will continue to denote the replacement space by $X$. Then we get a fibration

$$\begin{array}{ccc} F & \xrightarrow{i} & X \\ & & \big\downarrow p \\ & & K(A,n) \end{array} \tag{4}$$

where $F$ is $n$-connected and $i_*$ is an isomorphism on $\pi_i$ for $i > n$. So $F$ captures all the homotopy groups of $X$ above dimension $n$.

We now consider odd dimensional spheres, $S^{2n-1}$. We construct the fibration as in (4) for $S^{2n-1}$.

$$
\begin{array}{ccc}
F_{2n} & \xrightarrow{i} & S^{2n-1} \\
 & & \big\downarrow p \\
 & & K(\mathbf{Z}, 2n-1)
\end{array}
\tag{5}
$$

**Lemma 3.1.** *$H_i(F_{2n})$ is finite for all $i$.*

*Proof.* We will consider the homology Serre Spectral Sequence for the fibration (5). Note that $H_{2n-1}$ of the base is $\mathbf{Z}$, the lower homology groups are 0 and the higher homology groups are finite.

Note also that $F_{2n}$ is finite type by Propositon 1.2, and has zero homology below dimension $2n$. Assume inductively that $H_i(F_{2n})$ is finite for $i < m$, and of course we only need consider $m \geq 2n$.

$$H_m(F_{2n}) = E^2_{0,m}$$

Note first that $E^2_{p,q}$ for $q < m$ is finite except for $(0,0)$ and $(2n-1,0)$. This is because

$$E^2_{p,q} = H_pK(\mathbf{Z}, 2n-1) \otimes H_qF_{2n}$$

and both groups are finitely generated, and at least one is finite for $q < m$ except in the two cases indicated.

We next note that $E^\infty_{0,m} = 0$ because $H_m(S^{2n-1}) = 0$ (we are using that $m \geq 2n$ of course). So we examine the stages between $E^2_{0,m}$ and $E^\infty_{0,m}$.

$$
\begin{aligned}
E^3_{0,m} &= \operatorname{cok}(d^2_{2,m-1} : E^2_{2,m-1} \to E^2_{0,m}) \\
E^4_{0,m} &= \operatorname{cok}(d^3_{3,m-2} : E^3_{3,m-2} \to E^3_{0,m}) \\
&\vdots \\
E^{m+2}_{0,m} &= \operatorname{cok}(d^{m+1}_{m+1,0} : E^{m+1}_{m+1,0} \to E^{m+1}_{0,m}) \\
E^\infty_{0,m} &= E^{m+2}_{0,m} = 0.
\end{aligned}
$$

In each line, the source of the differential is a finite group - that is we've avoided our trouble spot for dimensional reasons or by inductive hypothesis. So by inducting back up the sequence,

$$E^{m+1}_{0,m}, E^m_{0,m}, \ldots E^2_{0,m}$$

are finite groups. Thus $H_m(F_{2n})$ is finite. $\square$

**Remark:** In particular,

$$\pi_{2n}(S^{2n-1}) = \pi_{2n}(F_{2n}) = H_{2n}(F_{2n})$$

is finite. This suggests something a little different happens for even spheres since you already know $\pi_3(S^2) = \mathbf{Z}$.

We want to continue this process so we can know about the rest of the homotopy groups of $S^{2n-1}$. Define $F_m$ inductively for $m > 2n$ by

$$
\begin{array}{ccc}
F_m & \xrightarrow{i} & F_{m-1} \\
 & & \big\downarrow p \\
 & & K(\pi_{m-1}F_{m-1}, m-1)
\end{array}
\tag{6}
$$

Here $m-1 \geq 2n$ so we can get started. The map $p$ is an isomorphism on $\pi_{m-1}$. By induction, $F_m$ is $m-1$-connected and $i_*$ is an isomorphism on $\pi_n$, $n \geq m$.

**Lemma 3.2.** *For all $m \geq 2n$, $H_i F_m$ is finite for each $i$, and $F_m$ is $m-1$ connected.*

*Proof.* We induct. $m = 2n$ is the base case, and that was Lemma 3.1.

Assume for $m-1 \geq 2n$. Then $\pi_{m-1}F_{m-1} = H_{m-1}F_{m-1}$ which is finite. The fact that $p$ is an isomorphism on $\pi_{m-1}$ with the long exact sequence on homotopy tells us that $F_m$ is $m-1$-connected.

By induction assumption, each homology group of $F_{m-1}$ is finite. By Corollary 2.4 each homology group of $K(\pi_{m-1}F_{m-1}, m-1)$ is finite. So by Proposition 1.4 applied to the class of finite abelian groups, each homology group of the fiber is finite. □

**Theorem 3.3.** *$\pi_i(S^{2n-1})$ is finite for all $i > 2n-1$.*

*Proof.* Let $i > 2n-1$. By iterating the maps $i$ from (5) and (6), $\pi_i F_i = \pi_i S^{2n-1}$. By Lemma 3.2, $\pi_i F_i = H_i F_i$ and is finite. □

Next we consider $S^{2n}$. We know the situation is more interesting here because the Whitehead square of the identity gives an element of Hopf invariant 2 (and thus infinite order) in $\pi_{4n-1}S^{2n}$. What we want to show is that apart from the groups generated by this element and the identity, the other homotopy of $S^{2n}$ is finite.

Here is a sketch of how to deal with this case. We use the same notation, $F_m$ for the "covers" of $S^{2n}$, so that $F_m, m > 2n$ is $m-1$ connected and $\pi_i F_m = \pi_i S^{2n}$ for $i \geq m$.

(1) Note that all $F_m$ are locally finite by iterating Proposition 1.2.
(2) Do the SSS for rational homology of $F_{2n+1} \to S^{2n} \to K(\mathbf{Z}, 2n)$ to conclude that

$$\tilde{H}_*(F_{2n+1}; \mathbf{Q}) = \begin{cases} \mathbf{Q} & * = 4n-1 \\ 0 & \text{else} \end{cases}$$

(3) The two facts above imply $H_i F_{2n+1}$ is finite except in dimensions 0 and $4n-1$ where there is a finitely generated group with a single infinite cyclic summand.
(4) Use this and induction up the tower with the SSS to conclude that each homology group of $F_i$ is finite except in dimensions 0 and $4n-1$ where there is a finitely generated group with a single infinite cyclic summand, for $2n+1 \leq i \leq 4n-1$.
(5) Use this to conclude that $H_i F_i = \pi_i F_i = \pi_i S^{2n}$ is finite for $2n+1 \leq i \leq 4n-2$ and that $\pi_{4n-1}F_{4n-1} = \pi_{4n-1}S^{2n}$ is infinite cyclic direct sum some finite group.
(6) Use another SSS argument to conclude that each reduced homology group of $F_{4n}$ is finite.
(7) Conclude that each reduced homology group of $F_m$ is finite for $m \geq 4n$ and thus $\pi_m S^{2n} = \pi_m F_m = H_m F_m$ is finite.

There is another much more elegant way to deal with the even dimensional spheres, but it involves some more technology (not spectral sequences). Here is an outline of the more elegant approach.

(1) The James construction on a pointed space $(X, *)$ is the "free monoid" on $X$. That is
$$JX = (X \sqcup X \times X \sqcup X \times X \times X \sqcup \cdots)/\sim$$
where $(x_1, \ldots, x_k) \sim (x_1, \ldots, x_i, *, x_{i+1}, \ldots, x_k)$. So the relation allows one to compare sequences of different lengths by inserting (or deleting) basepoints. This is a monoid under concatenation, with unit $(*)$.

(2) There is a map $JX \to \Omega\Sigma X$ where we use the reduced suspension and Moore loops. This is a map of monoids. $*$ goes to the constant loop of length 0. $x$ goes to the loop $t \mapsto (t, x)$.

(3) This map is a weak equivalence and when $X$ is a CW complex, this map is a homotopy equivalence.

(4) One can prove that
$$\Sigma JX = \bigvee_{i=1}^{\infty} \Sigma(X^{\wedge i})$$
which implies the same for $\Sigma\Omega\Sigma X$ when $X$ is a CW complex.

(5) Using this decomposition, we can make a map known as the Hopf map
$$\Sigma\Omega S^{2n} = \Sigma J S^{2n-1} \to \Sigma S^{4n-2} = S^{4n-1}$$
which is just projection out to the sphere $\Sigma(S^{2n-1} \wedge S^{2n-1})$.

(6) If we take the adjoint of this map we get a map
$$\Omega S^{2n} \xrightarrow{h} \Omega S^{4n-1}.$$
This induces isomorphism on $H_i$ when $i$ is a multiple of $4n-2$. (To prove this one has to calculate both cohomology rings, and note that we get an isomorphism in dimension $4n-2$).

(7) Let $F$ be the homotopy fiber of that map and look at the fibration
$$\begin{array}{ccc} F & \longrightarrow & \Omega S^{2n} \\ & & \big\downarrow h \\ & & \Omega S^{4n-1}. \end{array}$$

(8) The fact that $h_*$ is onto in homology tells us that none of the classes in $E^r_{p,0}$ support differentials, and one can use this fact to show that the SSS collapses at $E^2$ and that $H_*(F) \cong H_* S^{2n-1}$. Since $F$ is simply connected, this is enough to see that there is a map $S^{2n-1} \to F$ which is a weak equivalence.

(9) This means there is a long exact sequence in homotopy groups associated to $S^{2n-1} \to \Omega S^{2n} \to \Omega S^{4n-1}$.

Since we know $\pi_* S^{2n-1}$ is finite except in dimension $2n-1$, and $\pi_*\Omega S^{4n-1}$ is finite except in dimension $4n-2$, we see that $\pi_*\Omega S^{2n}$ is finite except in dimensions $2n-1$ and $4n-2$, and in each of those dimensions it is the direct sum of $\mathbf{Z}$ with a finite group.

(10) To get the same results for $\pi_* S^{2n}$, shift everything up one dimension.