Spectral Sequence Notes: Local coefficients and fibrations

1. Local coefficient systems

Definition 1.1. *The* fundamental groupoid *of a space X is a category denoted $\pi(X)$ where:*

$$ob(\pi(X)) := X$$
$$Hom(p,q) := \{\omega : I \to X : \omega(0) = p, \omega(1) = q\}/\sim$$

where the relation $\sim$ is path homotopy.

Notice that as the definition suggests, this is a *groupoid*, that is a category in which every morphism is invertible (simply take the reversed path for the inverse of a path).

Also notice that $\mathrm{Hom}(p,p) = \pi_1(X,p)$. This category is simply a way of organizing all the fundamental groups of X and all the isomorhisms between them induced by paths.

Definition 1.2. *A* system of local coefficients *is a contravariant functor*

$$\mathcal{G} : \pi(X) \to \underline{AbGps}.$$

This is also called a *bundle of abelian groups*, a *system of local coefficients*, just *local coefficients* or a *local system*, and also *twisted coefficients.*

If X is locally one-connected (locally path connected and locally simply connected) $\mathcal{G}$ can be used to derive a covering space, but not generally a connected covering space. In fact it will have a global section given by taking X to the identity element in $\mathcal{G}(x)$. It is also a locally constant sheaf of discrete groups.

Caution: it is not necessarily a principal bundle since the structure group can be as big as all group automorphisms of the fiber, and it is also not an arbitrary fiber bundle since the structure group isn't as big as all set bijections of the fiber, plus it has a global section.

Example 1.3. *Let A be a fixed abelian group. Define $\mathcal{G} : \pi(X) \to \underline{AbGps}$ by $\mathcal{G}(p) = A$ for all $p \in X$ and $\mathcal{G}([\omega]) = 1_A$ for all paths ω. This is the constant system of local coefficients.*

Example 1.4. *Define*

$$\mathcal{G} : \pi(X) \to \underline{Gps}$$

by

$$\mathcal{G}(p) := \pi_1(X,p)$$
$$\mathcal{G}(\omega) := \omega_\# : \pi_1(X,\omega(1)) \to \pi_1(X,\omega(0)).$$

Here $\omega_\#$ is the usual map defined on a loop λ by

$$\omega_\#([\lambda]) = [\omega * \lambda * \omega^{-1}].$$

It is a first year algebra topology exercise to show that $\omega_\#$ is independent on the representative λ and depends only on the path homotopy class of ω, and hence that it is an isomorphism with inverse $(\omega^{-1})_\#$.

Note that Example 1.4 is not necessarily a system of local coefficients since it takes values $\underline{Gps}$ rather than $\underline{AbGps}$. But it is such a fundamental example that we give it anyway with the remark that we only get a system of local coefficients if the fundamental group of X happens to be abelian.

Definition 1.5. *Let $\mathcal{A}$ be a system of local coefficients on a space X.*

$$C_p(X, \mathcal{A}) = \{\Sigma g_i \otimes \sigma_i : \sigma_i : \Delta^p \to X, g_i \in \mathcal{A}(\sigma_i(v_0))\} \subseteq \bigoplus_{p \in X} \mathcal{A}(p) \otimes C_p(X).$$

$$d_p : C_p(X, \mathcal{A}) \to C_{p-1}(X, \mathcal{A})$$

is given on generators by

$$d_p(g \otimes \sigma) = \sum_{i=1}^{p} (-1)^i g \otimes \delta_i \sigma + \mathcal{A}(\lambda_\sigma)(g) \otimes \delta_0 \sigma.$$

To explain the notation in the formula for d_p: $\delta_i\sigma$ is as usual the $p-1$ simplex defined by restricting σ to the face across from the ith vertex. Note that $\delta_j\sigma$ has the same value on v_0 as σ when $j \geq 1$. But $\delta_0\sigma$ takes v_0 to $\sigma(v_1)$ so the element $g \in \mathcal{A}(\sigma(v_0))$ is no longer an allowable coefficient.

λ_σ is the path from $\sigma(v_1)$ to $\sigma(v_0)$ given by

$$t \mapsto \sigma(tv_0 + (1-t)v_1).$$

And then of course

$$\mathcal{A}(\lambda_\sigma) : \mathcal{A}(\sigma(v_0)) \to \mathcal{A}(\sigma(v_1)).$$

Lemma 1.6. *Using the formula in Definition 1.5, $d_{p-1}d_p = 0$.*

This is essentially the usual proof for singular homology. One just has to attend to what is going on with the change of groups, and realize that any paths gotten from applying σ to a path from v_2 to v_0 are homotopic to any other such paths.

Definition 1.7. *Of course now we have the obvious definition:*

$$H_p(X; \mathcal{A}) := H_p C_*(X; \mathcal{A}).$$

Note that there is a cohomology with local coefficients too.

$$C^p(X, \mathcal{A}) := \{c : C_p(X) \to \oplus_{x \in X} \mathcal{A}(x) : c(\sigma) \in \mathcal{A}(\sigma(v_0))\}.$$

One has to make a similar adjustment to the coboundary map, and then one can define $H^*(X; \mathcal{A})$. This is also a special case of sheaf cohomology.

The next question is when and how we can compute $H_*(X; \mathcal{A})$. The first observation is that if $\pi(X)$ is trivial enough, this reduces to ordinary homology.

Suppose X is path connected, and $\pi_1(X, x_0) = 0$. Then by a standard algebraic topology exercise, given $p, q \in X$, there is only one path homotopy class of paths starting at p and ending at q. So in $\pi(X)$, there is a unique morphism p to q, and of course it also has a unique inverse.

In this situation, if $\mathcal{A}$ is a local system on X, then there is a unique isomorphism $f_{p,q} : \mathcal{A}(p) \to \mathcal{A}(q)$ for any $p, q \in X$. Choose $x_0 \in X$ as basepoint and write A for $\mathcal{A}(x_0)$. We get an isomorphism of chain complexes

$$\begin{array}{ccc} C_p(X; \mathcal{A}) & \xrightarrow{U_p} & C_p(X; A) \\ {\scriptstyle d_p}\big\downarrow & & \big\downarrow{\scriptstyle d} \\ C_{p-1}(X; \mathcal{A}) & \xrightarrow{U_{p-1}} & C_{p-1}(X; A) \end{array}$$

by sending $g \otimes \sigma$ to $f_{\sigma(0), x_0}(g) \otimes \sigma$.

Lemma 1.8. *The map U_* gives an isomorphism of chain complexes.*

Corollary 1.9. *If X is one-connected and $\mathcal{A}$ is a local system of coeffficients, and $A = \mathcal{A}(x_0)$ for some point $x_0 \in X$ then*

$$H_*(X; \mathcal{A}) = H_*(X; A).$$

It is worth noting that we may be able to relax the $\pi_1(X, x_0) = 0$ condition. All we've really used is that in $\mathcal{A}$ there is a unique isomorphism from $\mathcal{A}(p)$ to $\mathcal{A}(q)$ for all p, q. Here are other conditions under which that is guaranteed.

- If some $\mathcal{A}(p)$ is isomorphic to 0 (and thus all are isomorphic to 0).
- If some $\mathcal{A}(p)$ is isomorphic to $\mathbf{Z}/(2)$ (and thus all are isomorphic to $\mathbf{Z}/(2)$.
- If some $\pi_1(X, x_0)$ is finite with odd order, and some $\mathcal{A}(p)$ is isomorphic to $\mathbf{Z}$ or $\mathbf{Q}$ or $\mathbf{R}$. Consideration of how the fundamental group acts on $\mathcal{A}(x_0)$ will lead you to realize that we have the "unique isomorphism" condition under this hypothesis.
- More generally if $\pi_1(X, x_0)$ is finite and torsion subgroup of the automorphism group of $\mathcal{A}(p)$ has order prime to that of $\pi_1(X, x_0)$.

All the situation listed immediately above (and in Corollary 1.9) are situations where the local system of coefficients is equivalent to a constant local system of coefficients. We still have a chance of computing this even if we don't know that.

Assume X is path connected. Let S be a set on which $\pi_1(X, x_0)$ acts. Then by the usual covering space techniques, we can make a covering space

$$\tilde{X} \xrightarrow{\pi} X$$

with $\pi^{-1}(x_0) = S$. Note that $\pi_1(X, x_0)$ may or may not act transitively on S, and the space $\tilde{X}$ may or may not be connected.

We get a system of local coefficients, $\mathcal{S}$ as follows:

$$\begin{aligned} \mathcal{S}(x) &:= \mathbf{Z}[\pi^{-1}(x)] \\ \mathcal{S}([\omega]) &:= \mathbf{Z}[\omega_{\#}^{-1}] : \mathbf{Z}[\pi^{-1}(\omega(1))] \to \mathbf{Z}[\pi^{-1}(\omega(0))] \end{aligned}$$

In the notation above, $\mathbf{Z}[S]$ denotes the free abelian group with generators S. $\omega_{\#}$ is the isomorphism from $\pi^{-1}(\omega(0))$ to $\pi^{-1}(\omega(1))$ induced by lifting the inclusion $\pi^{-1}(\omega(0)) \subseteq \tilde{X}$ along ω.

Proposition 1.10. *Under the hypotheses above,*

$$H_*(X; \mathcal{S}) = H_*(\tilde{X}; \mathbf{Z}).$$

In the situation we've just described, we could use another ring R in place of $\mathbf{Z}$. The most common choice would be some field k.

2. Fibrations and bundles of groups corresponding to the fiber

Assume

$$\begin{tikzcd} F \arrow[r, "i"] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

is a fibration. Let ω be a path in B from $p = \omega(0)$ to $q = \omega(1)$. Let $F_p = \pi^{-1}(p)$. Consider the diagram

$$\begin{array}{ccc} F_q \times 1 & \xrightarrow[i]{} & E \\ \downarrow & & \downarrow p \\ F_q \times I & \xrightarrow[\omega]{} & B \end{array}$$

where the top horizontal map is the obvious inclusion of the fiber F_q and the bottom horizontal map is $(f,t) \mapsto \omega(t)$. Because p is a fibration, there is a map $G : F_q \times I \to E$ making the diagram commute.

$$\omega_{\#} = G|_{F_q \times \{0\}} : F_q \to F_p.$$

It is a first year algebraic topology exercise to check that $\omega_{\#}$ is a homotopy equivalence and that in fact the homotopy class of $\omega_{\#}$ depends only on the path homotopy class of ω, and not on either the representative, or the lift G. Let A be an abelian group. We've outlined the proof that the association

$$\begin{aligned} b &\mapsto H_i(F_b; A) \\ [\omega] &\mapsto H_i(\omega_{\#}) \end{aligned}$$

gives a system of local coefficients which we'll call $\mathcal{H}_i(F; A)$.

Note that we could use other functors in place of H_i. We could use, for example, $H^i(F_b; A)$ though we need to use $\omega_{\#}^{-1}$ to adjust the variance. Or we could use $\pi_i(F_b)$ if F is simple, so that $\pi_i(F)$ doesn't depend on the basepoint chosen.

Proposition 2.1. *Let A be an abelian group. If F, B are path connected and $\pi_1(B) = 0$ then the E^2 term of the homology and cohomology Serre spectral sequences are given (respectively) by*

$$\begin{aligned} E^2_{p,q} &= H_p(B; H_q(F; A)) \\ E_2^{p,q} &= H^p(B; H^q(F; A)) \end{aligned}$$

Proof: Apply Proposition 1.10 to Serre's result that $E^2_{p,q} = H_p(B; \mathcal{H}_q(F; A)$. The same idea works for cohomology.

Proposition 2.2. (1) *Assume the hypotheses of Proposition 2.1. Assume in addition that either A is a field, or it is a commutative ring and either $H_p(B; A)$ is free over A or $H_q(F; A)$ is free over A. Then*

$$E^2_{p,q} = H_p(B; A) \otimes_A H_q(F; A).$$

(2) *Assume the hypotheses of Proposition 2.1. Assume in addition that either A is a field, or it is a commutative ring and either $H^p(B; A)$ is free over A or $H^q(F; A)$ is free over A, and one of those two groups is finitely generated over A. Then*

$$E_2^{p,q} = H^p(B; A) \otimes_A H^q(F; A).$$

3. Potential Exercises

(1) Prove that $\pi(X)$ is a groupoid. Then prove that if X is simply connected, there is a unique isomorphism from each $p \in X$ to each $q \in X$.
(2) Prove Lemma 1.6.
(3) Define the coboundary on $C^p(X; \mathcal{A})$ and prove that it is a differential (that is squares to 0).
(4) Prove Lemma 1.8.
(5) Let X be path connected. Suppose $\pi_1(X, x_0) \cong \mathbf{Z}/(p)$. Let $\mathcal{A}$ be a local system of coefficients satisfying $\mathcal{A}(p) \cong \mathbf{Z}$ for all p. Show that this is (isomorphic to) a constant local system of coefficients.
(6) Let X be path connected. Suppose $\pi_1(X, x_0) \cong G$ for some finite group G. Let $\mathcal{A}$ be a local system of coefficients satisfying $\mathcal{A}(p) \cong A$ for all p. Assume that every group automorphism of A has order prime to $|G|$ or infinite order. Show that $\mathcal{A}$ is (isomorphic to) a constant local system of coefficients.
(7) Prove Proposition 1.10. You'll want to do this by creating an isomorphism of chain complexes, and that will require using the lifting property for covering spaces.
(8) Find an example so that $\tilde{H}_*(X; \mathbf{Z}) = 0$ but $\tilde{H}_*(X; \mathcal{A}) \neq 0$.
(9) Find an example so that $\tilde{H}_*(X; \mathbf{Z}) \neq 0$ but $\tilde{H}_*(X; \mathcal{A}) = 0$.