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Asset price volatility in a nonconvex general equilibrium model★

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Summary. Asset prices and returns are known to vary significantly more than output or aggregate consumption growth, and an order of magnitude in excess of what is justified by innovations to fundamentals. We study excess price volatility in a lifecycle economy with two assets (claims on capital and a public debt bubble), heterogeneous agents, and increasing returns to financial intermediation. We show that a relatively modest nonconvexity generates a set valued equilibrium correspondence in asset prices, with two stable branches. Price volatility is the outcome of an equilibrium selection mechanism, which mixes adaptive learning with "noise", and alternates stochastically between the two stable branches of the price correspondence.

Keywords and Phrases: Private information, Costly state verification, Asset price volatility, Cycles.

JEL Classification Numbers: E32, E44, G12, G14.

1 Introduction

Twenty years after LeRoy and Porter (1981) and Shiller (1981) presented the first evidence of excess volatility in annual U.S. stockmarket indices, a wealth of evidence has come in to bolster the view that asset prices and asset returns vary too much relative to recorded dividend streams (West L, 1988; Campbell and Kyle, 1993) and in comparison to changes in aggregate income or consumption. For example, quarterly detrended U.S. output has standard deviation 0.0167, about one-fifth of the corresponding 0.0820 figure for

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stockmarket returns[1]. The standard deviation of 0.006 in detrended annual dividends is a small fraction of the corresponding deviation of 0.174 in the stockprice index[2]. Econometric studies of stock prices and returns show that only a fraction, typically 5% to 20% , of asset price innovations can be attributed to innovations in dividends and interest rates[3].

Because aggregate time series for consumption and dividends are smooth, consumption-based asset pricing models with a representative household, like Lucas (1978), can account for relatively abrupt year-to-year price changes only by assuming unrealistically high risk-aversion or unrealistically low intertemporal elasticities of substitution. To explain excess volatility, as the term is understood in the finance literature, asset prices should contain a bubble component. On the other hand, excess volatility in the macroeconomic or general equilibrium sense requires price movements uncorrelated with economic fundamentals like tastes, technology or endowments.

We capture these two alternative definitions of excess volatility in an economy with two assets, a claim on capital that may exhibit excess volatility in the macroeconomic sense, and a public liability that satisfies the finance definition of the same term. Section 2 of this paper embeds these assets in a general equilibrium model with two-period lived overlapping generations and heterogenous households. Section 3 shows that a reasonable degree of increasing returns in financial intermediation results in an equilibrium correspondence with two stable branches. In section 4, we present an equilibrium selection mechanism mixing adaptive learning with random impulses by "noise" traders, which oscillates between these two branches, creating nonfundamental volatility in asset prices. Farmer (1997) provides a related explanation of asset price volatility in an economy in which multiple an equilibria stem from a positively valued bubble in an overlapping generations economy with a potentially infinite lifecycle. In our paper, volatility arises instead from the set-valuedness of the equilibrium correspondence, and may persist even without bubbles.

2 The economy

2.1 Private agents

The economy is inhabited by a continuum of two-period lived overlapping generations of economic agents. Each generation has unit mass (hence total population is constant), and consists of two types of agents.

Households, comprising $\lambda \in (0, 1)$ fraction of the population, consume in both periods of life. Their preferences over youth and old-age consumption is given by the logarithmic utility function:

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[1] Reported in Cooley and Quadrini (1998) from H-P filtered data, 1959:01–1996:04. Baxter (1991) reports linearly detrended quarterly output to have volatility of about 0.04.

[2] See Campbell and Kyle (1993).

[3] See West, and Campbell and Kyle, op. cit.

$$U^H(c_t^t, c_{t+1}^t) = \log c_t^t + \beta \log c_{t+1}^t, \quad \beta > 0 \ , \tag{1}$$

where a superscript denotes the generation, and a subscript denotes calendar time. Households do not own any capital or final goods[4], but they consume out of wage income earned in youth from supplying inelastically 1 unit of labor time. Agents endowed with the logarithmic utility function (1) save a constant fraction, $s = \beta/(1+\beta)$, of their first period income.

Savings made by young households have two alternative uses. They may be deposited in banks and/or used to hold a government-issued asset, which we shall call bonds (see below). Savings deposited with banks in period t are paid a sure return R_{t+1}^D the following period, on each unit deposited. The government asset also pays the same sure return on each unit of bond held from period t to $t+1$. Therefore, from the household's point of view, the two assets are perfect substitutes.

The remaining fraction, $1-\lambda$, of the population consists of investment goods producers or *investors*. They are risk-neutral, and consume only in old age, so that their utility function is simply:

$$U^I(c_t^t, c_{t+1}^t) = c_{t+1}^t \ .$$

Each investor has *no* endowment of time or goods, but owns an investment technology. In period t, a young investor i can borrow b_t units of resources and convert them into capital in period $t+1$ according to the stochastic constant returns technology:

$$k_{t+1} = \theta^i b_t \ . \tag{2}$$

θ^i is a private shock distributed independently and identically across investors, with mean one and a cumulative distribution function $H(\theta^i)$ on the bounded support $\Theta \equiv [\underline{\theta}, \overline{\theta}]$. Each investor observes her idiosyncratic shock soon after borrowing (in the same period), but the capital is produced only at the beginning of the following period. This capital is rented out to firms; consumption and loan repayments are then made out of rental income. We note that independence and unit mean assumptions about θ^i imply that on average the economy produces as much capital as the amount loaned out the previous period.

2.2 Firms

Firms hire the labor services of young households and rent capital from investors to produce final goods according to the constant returns to scale production technology $F(K, N)$. Labor and capital markets are perfectly competitive, so that both factors of production earn their marginal products. We shall henceforth ignore sales of undepreciated capital to younger agents by assuming that the depreciation rate is $\delta = 1$.

[4] The initial old generation of households has an aggregate capital endowment of $K_0 > 0$.

2.3 The government

The role of the government in this economy is limited to rolling over public debt; the government does not consume goods or services, nor does it invest. Every period it floats new bonds, maturing next period, on the market to refinance public debt outstanding from the previous period. We shall abstract from any kind of default risk on government bonds. Hence the return on public debt is riskless and, in particular, the same as the return on bank deposits. Denoting d_t as the per capita government bonds maturing in period t, the evolution of the stock of per capita bonds is governed by the equation:

$$d_{t+1} = R_t^D d_t \quad . \tag{3}$$

2.4 Banks

Banks intermediate all borrowing and lending activities in this economy. They take in deposits from young households and give out loans for capital investment to young investors. This intermediation is an optimal arrangement, as shown in Williamson (1986), since there is a potential moral hazard problem in the credit market. The bank is able to exploit the law of large numbers by interacting with many borrowers and depositors, predicting with certainty the fraction of investments that have bad outcomes, and guaranteeing a sure gross return of R_{t+1}^D on deposits made in period t. Banks enter into contracts with investors, taking as given the return that has to be paid on deposits. However, because of private information, they are unable to costlessly observe the returns on these investment projects.

2.5 Financial intermediation and loan contracts[5]

Borrowing and lending activities between a bank and an investor are governed by the terms of a loan contract. Banks offer loan contracts to each potential borrower (investor) and in doing so, maximize their profits subject to an incentive compatibility constraint and an individual rationality (or participation) constraint for each borrower. The optimal loan contract is obtained as a solution to this principal-agent problem, with Nash competition among banks driving their equilibrium profits to zero.

With costly state verification and deterministic monitoring, Gale and Hellwig (1985) show that the optimal loan contract is a standard debt contract. Hence, the contract δ can be characterized by the triple $(b, x, R^L) \in R_+^3$ which specifies the loan size b, a critical value for the idiosyncratic state, $x \in \Theta$, and the gross yield R^L owed by the investor to the bank, on each unit borrowed. It is important that the contract should specify a critical state x, below which verification occurs for sure, to prevent borrowers from defaulting in every state.

[5] This section draws on Azariadis and Chakraborty (1998).

Each contract is executed in the following way. If an investor's realized idiosyncratic shock is less than x, she is unable to repay bR^L and declares bankruptcy. The bank pays a proportional cost of γ units of capital per unit loan[6] to verify that the state is indeed below x. It then takes over the bankrupt project and brings it to completion. The amount recovered from the bankrupt project is simply the one period rental income, $l(\theta) = \rho\theta b$, from whatever capital is produced, ρ being the rental rate. On the other hand, if the realized idiosyncratic shock is above x, the investor is solvent and pays back bR^L. For the contract to be incentive compatible the loan rate, R^L, has to be independent of the realized value of the shock. Morevoer, continuity of the payoff function requires that $R^L b = l(x) = \rho x b$. This simplifies solving for the contract as we now need to care only about the optimal values of b and x.

An investor's expected payoff from a contract δ is

$$U(\delta) = b\rho[1 - \mu(x)] \tag{4}$$

where $\mu(x) \equiv x[1 - H(x)] + \int_{\underline{\theta}}^{x} \theta dH$. The contract has to guarantee the investor a minimum utility level U_0 to ensure participation. In 'partial equilibrium', U_0 is taken as given by the bank, but is determined endogenously in general equilibrium by the volume of savings and provisions for agency costs in any period.

The verification cost $\gamma \in [0, \underline{\theta}]$ is assumed to be proportional to the loan size and hence, a bank's expected payoff is

$$\Pi(\delta, q) \equiv \rho b[M(x, \gamma) - q] \ , \tag{5}$$

where $q \equiv R^D/\rho$ and

$$M(x, \gamma) \equiv x[1 - H(x)] + \int_{\underline{\theta}}^{x} (\theta - \gamma) dH \ . \tag{6}$$

We also assume free entry into banking so that maximal bank profits are zero in equilibrium. This allows us to determine the deposit rate R^D, given the optimal loan contract as characterized by the following theorem; Appendix A provides details.

Theorem 1 *Given* (γ, ρ) *and the expected investor payoff* U_0, *the optimum loan contract* $\hat{\delta} = (\hat{b}, \hat{x}, \hat{R}^L)$ *satisfies*

i) $\hat{x} = \arg\max_{x \in \Theta} M(x, \gamma)$,

ii) $\widehat{R}^L = \rho \widehat{x}$,

iii) $\widehat{b} = U_0/\rho[1 - \mu(\hat{x}(\gamma))]$.

The first condition in Theorem 1 is simply the profit maximizing choice of the critical state x. Condition (ii) follows from our previous discussion of the optimum loan contract taking the form of a standard debt contract, while (iii) determines the loan size *given* U_0 and γ.

[6] We assume a committment device which rules out ex-post renegotiation between the investor and the bank.

3 General equilibrium

3.1 Intermediaries and asset markets

Since there is a continuum of investors, the expected value of the idiosyncratic shock is one and idiosyncratic shocks are uncorrelated across investors, it follows that the return per unit loan net of auditing costs is, $\rho M(x, \gamma)$, by the weak law of large numbers. Competition among banks drives maximal profits to zero so that the net return per unit loan equals the cost of loanable funds $\rho q \equiv R^D$.

The total inflow of funds into banks, deposits D_t, equals the outflow of funds, loans L_t, plus reserves for agency costs to be incurred on current loans. Thus

$$D_t = L_t + \gamma H(x_{t+1}) L_t \Rightarrow L_t = \frac{D_t}{1 + \gamma H(x_{t+1})} \tag{7}$$

where $x_{t+1} = \hat{x}(\gamma)$. Figure 1 illustrates the time line of bank operations. At the beginning of period t, capital from period $t-1$ projects is produced, and rented out to firms. Wages are paid out to young period t households, which are then deposited with banks and held in government bonds. Given the volume of deposits, banks make provisions for agency costs, anticipating how much verification costs they will have to incur on insolvent projects later in period t. All remaining funds are then loaned out to young investors. Each investor observes her shock soon after receiving loans, and if the realized value is less than x, declares bankruptcy at once. The bank takes over the bankrupt project, and recovers part of its costs from the rental income of the capital it produces.

Total deposits D_t are given by total savings by young households, $sw(k_t)$, net of bond holdings, d_{t+1}. Therefore total loans are

$$L_t = \frac{sw(k_t) - d_{t+1}}{1 + \gamma H(\hat{x}(\gamma))} \quad .$$

Since $E(\theta) = 1$, the law of large numbers implies that the capital stock in period $t+1$ is equal to loans made in period t, $k_{t+1} = L_t$.

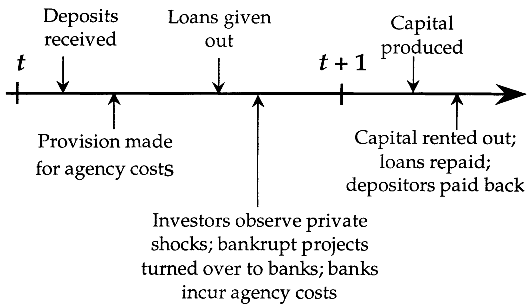

Figure 1. Timeline of bank operations

Equilibrium sequences of the capital stock satisfy the following first order difference equation:

$$k_{t+1} = \frac{sw(k_t) - d_{t+1}}{1 + \gamma H(\hat{x}(\gamma))} = \frac{sw(k_t) - d_{t+1}}{1 + \phi(\gamma)} \quad , \tag{8}$$

where

$$\phi(\gamma) = H[\hat{x}(\gamma)] \tag{9}$$

is the fraction of bankrupt projects.

General equilibrium is described by the two difference equations, (3) and (8), which constrain the economy to satisfy the government budget identity and the asset market clearing condition. Before looking at the stability properties of stationary equilibria in this economy, it will be helpful to make some simplifying assumptions about the investment and production technologies. The idiosyncratic stochastic shock θ is assumed to be drawn, *i.i.d.*, from a uniform distribution with compact support $[1 - \varepsilon, 1 + \varepsilon]$. The production function is assumed to be Cobb-Douglas with $F(K, N) = K^{\alpha} N^{1-\alpha}$ and $0 < \alpha < 1$.

Our explanation of asset price volatility depends crucially on the assumption of external increasing returns to scale in financial intermediation, an assumption that agrees with the observed countercyclical movement in the gap between borrowing and lending rates. Azariadis and Chakraborty (1998) explore how this kind of nonconvexity can explain the propagation and amplification of shocks at business cycle frequencies; an informational indivisibility plays the same role in Azariadis and Smith (1998). Each bank's cost of state verification per unit loan is assumed henceforth to be a decreasing function of activity at the industry level; no individual bank can influence unit costs on its own. Current unit agency costs will then depend upon the *aggregate* volume of current loans, that is, on *tomorrow's capital stock*. In other words, we have in mind a verification technology of the type $\gamma_t = \gamma(L_t) = \gamma(k_{t+1})$,where γ is a decreasing function of aggregate loans (hence of future capital stock). For tractability, we shall work with the case where γ is the following step function:

$$\gamma_t = \gamma(k_{t+1}) = \begin{cases} \gamma_0, & \text{if } k_{t+1} \leq k_c \\ 0, & \text{if } k_{t+1} > k_c \end{cases} \quad , \tag{10}$$

for some critical capital stock k_c and some γ_0 which satisfies $0 < \gamma_0 < \varepsilon$.

Under these assumptions, it is easy to show that:

$$\begin{aligned} \widehat{x}(\gamma) &= 1 + \varepsilon - \gamma \\ M[\widehat{x}, \gamma] &= 1 - \gamma + \gamma^2/4\varepsilon \quad , \end{aligned}$$

and that agency costs per unit loan $\gamma H[\widehat{x}(\gamma)]$, are increasing in the unit cost of verification, γ. Furthermore, the deposit and lending rates are

$$R_t^D = \rho_t \big(1 - \gamma_t + \gamma_t^2/4\varepsilon\big) = d_t/d_{t+1}, \quad R_t^L = \rho_t \widehat{x}(\gamma_t) \tag{11}$$

From equation (11) it is clear that the interest rate gap

$$R_t^L/R_t^D = (1-\varepsilon-\gamma_t)/(1-\gamma_t+\gamma_t^2/4\varepsilon) \tag{12}$$

is increasing in γ and hence, a decreasing function of the capital stock k. This pattern agrees with the typical countercyclical change of 200 basis points in the difference between the commercial paper rate and the yield on U.S. government bonds.

Combining these equations with (3) and (8), we conclude that competitive equilibria are solutions to the following dynamical system in the state variables (k_t, d_t):

$$k_{t+1} = B(\gamma,\varepsilon)[s(1-\alpha)k_t^\alpha - d_{t+1}] \tag{13}$$

$$d_{t+1} = \alpha M(\gamma,\varepsilon)k_t^{\alpha-1}d_t \tag{14}$$

where, $B(\gamma,\varepsilon) \equiv 1/[1+\gamma(1-\gamma/2\varepsilon)] \leq 1$ and $M(\gamma,\varepsilon) \equiv 1-\gamma+\gamma^2/4\varepsilon \leq 1$. Any solution to this system specifies completely the returns (ρ_t, R_t^D) to capital and debt and, by extension, the prices $(1/\rho_t, 1/R_t^D)$ of capital and debt in terms of foregone consumption.

For any $\gamma \geq 0$, the system (13) and (14) generally has three stationary states: two "nonmonetary" or real states with $(k,d) = (0,0)$ and $(\widetilde{k}, 0)$, respectively; and one "monetary" or nominal state (k^*, d^*) with $k^* > 0$, and $d^* > 0$ if the economy is Samuelson[7]. Furthermore, the nominal steady state is a saddle, the trivial real steady state is a source, and the nontrivial real steady state is a sink[8]. Figure 2 depicts the phase diagram for the cost function γ defined in (10).

Equilibrium paths that converge to the non-trivial steady state $(0, \widetilde{k})$ for any fixed γ are associated with a zero price of public debt. To exclude this possibility, we focus in the sequel on solutions that remain *asymptotically* nominal and asset prices that remain bounded away from zero. These solutions correspond to the stable manifold of the nominal steady state (k^*, d^*) which we proceed to analyze below.

3.2 Dynamics of nominal equilibria

For any given cost, γ, of state verification, the stable manifold, or saddle path, leading to the nominal steady state (k^*, d^*) turns out to be a ray through the origin in any Cobb-Douglas economy with full depreciation of capital. To see this, we define a new state variable, $z = d/k$, to describe the public debt-to-capital ratio and reduce equations (13) and (14) to

$$k_{t+1} = B[s(1-\alpha) - \alpha M z_t]k_t^\alpha \ , \tag{15}$$

$$z_{t+1} = \frac{\alpha M z_t}{B[s(1-\alpha) - \alpha M z_t]} \ . \tag{16}$$

[7] This is satisfied if $s(1-\alpha)/\alpha > (1-\gamma+\gamma^2/4\varepsilon)(1+\gamma-\gamma^2/2\varepsilon)$.

[8] See Cass and Yaari (1967) , Tirole (1983) or Azariadis (1993) for a fuller dynamic analysis of overlapping generation models with public debt.

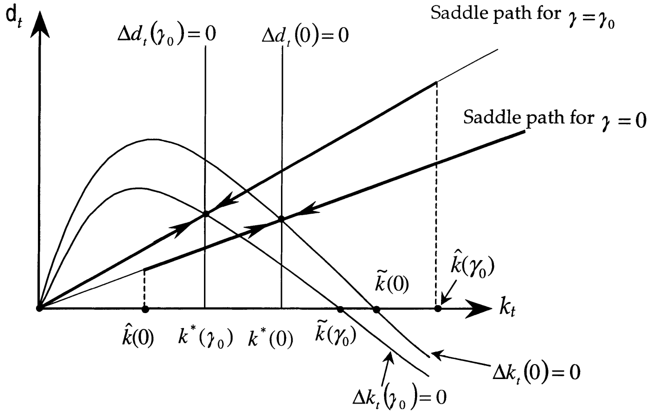

Figure 2. Phase diagram with nonconvex verification cost

Assuming[9] $s(1-\alpha) > \alpha M/B$, equations (15) and (16) admit three types of solution: one is *real equilibria* described as solutions to

$$z_t = 0, \qquad k_{t+1} = Bs(1-\alpha)k_t^{\alpha} \quad \forall t \ . \tag{17}$$

The second one is *nominal equilibria* which solve

$$z_t = [s(1-\alpha)/\alpha M - 1/B] \equiv z^*(\gamma) > 0, \qquad k_{t+1} = \alpha M k_t^{\alpha} \quad \forall t \ , \tag{18}$$

where M is defined by equation (10) to satisfy

$$M(\gamma_t) = \begin{cases} 1, & \text{if } \ k_{t+1} \geq k_c \\ 1 - \gamma_0 + \gamma_0^2/4\varepsilon \equiv M_0, & \text{if } \ k_{t+1} \leq k_c \end{cases}$$

Finally, *asymptotically real equilibria* solve (15) and (16) for initial values, z_0, of the debt-to-capital ratio that start below the positive state $z^*(\gamma)$. Formally, for any $z_0 \in (0, z^*)$, solutions to (15) and (16) converge to those of equation (17).

We focus exclusively on *nominal* equilibria. For any competitive allocation of this type, prices of capital and debt in terms of the consumption good are, respectively

$$p_t^K = 1/\rho_t = k_t^{\alpha}/\alpha, \qquad p_t^D = p_t^K/M > p_t^K \ . \tag{19}$$

Debt is naturally more expensive because it requires less information than capital; returns to debt need not be audited.

When returns to financial intermediation are increasing in the manner described by equation (10), investment lowers the cost of state verification and reduces the information advantage of public debt over physical capital. In a wide variety of circumstances, one may then expect that the steady state debt-to-capital ratio $z^*(\gamma)$ should be an *increasing* function of γ, the unit cost

[9] This inequality defines a Samuelson economy in the presence of auditing costs.

of state verification, because higher information costs would tend to favor public debt over capital in private asset portfolios.

For the economy analyzed in this paper, equation (18) means that

$$z^*(\gamma) = s(1-\alpha)/\alpha M - 1/B = s(1-\alpha)/\alpha\left(1+\gamma-\gamma^2/2\varepsilon\right) - [1+\gamma(1-\gamma/2\varepsilon)] \tag{20}$$

is easily shown to be an increasing function of γ for any $\gamma \in [0, \varepsilon]$. In particular, for the state verification cost function (10), nominal equilibria satisfy

$$\begin{array}{lll} z_t = z^*(\gamma_0), & k_{t+1} = \alpha M_0 k_t^\alpha, & \text{if } k_{t+1} \in [0, k_c] \\ z_t = z^*(0), & k_{t+1} = \alpha k_t^\alpha, & \text{if } k_{t+1} \in [k_c, \infty) \end{array} . \tag{21}$$

Equation (21) is equivalent to

$$\begin{array}{lll} z_t = z^*(\gamma_0), & k_{t+1} = \alpha M_0 k_t^\alpha, & \text{if } k_t \leq \widehat{k}(\gamma_0) \\ z_t = z^*(0), & k_{t+1} = \alpha k_t^\alpha, & \text{if } k_t \geq \widehat{k}(0) \end{array} , \tag{22}$$

where $\widehat{k}(\gamma_0)$ is an increasing function defined from the equality

$$\left[\widehat{k}(\gamma_0)\right]^\alpha = \frac{k_c}{[\alpha M(\gamma_0)]} . \tag{23}$$

Then, the steady states of the two branches of equation (22) satisfy

$$[k^*(\gamma_0)]^{1-\alpha} = \alpha M(\gamma_0), \quad [k^*(0)]^{1-\alpha} = \alpha . \tag{24}$$

The phase diagram for this economy in Figure 3 is set valued and equilibrium is indeterminate for values of the capital stock in the overlapping interval $k_t \in \left[\widehat{k}(0), \widehat{k}(\gamma_0)\right]$. Assuming that this interval contains the steady states of both branches of the equilibrium correspondence (22), that is, if we suppose

$$\widehat{k}(0) < k^*(\gamma_0) < k^*(0) < \widehat{k}(\gamma_0) , \tag{25}$$

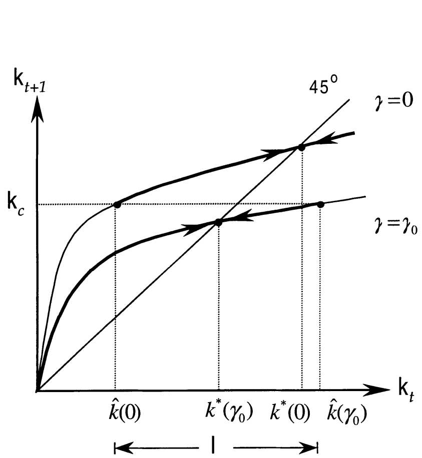

Figure 3. Capital dynamics with nonconvex verification cost

we are led to the conclusion that all equilibrium sequences will eventually converge to the invariant set

$$I = \{k \geq 0 \mid k^*(\gamma_0) \leq k \leq k^*(0)\}$$

which spans these two steady states.

The simplest way to analyze the behavior of equilibrium sequences over the invariant set is to express everything in terms of asset prices. Define $q_t = \log p_t^K$, the natural logarithm of the price of capital. Then take logs in equation (22) and obtain

$$q_{t+1} = \begin{cases} b_0 + \alpha q_t, & \text{if} \quad \gamma = \gamma_0 \\ b_1 + \alpha q_t, & \text{if} \quad \gamma = 0 \end{cases} \tag{26}$$

where,

$$b_0 \equiv A(1-\alpha)\log M_0 < A(1-\alpha)\log 1 = 0 \equiv b_1 \ , \tag{27}$$

and $A > 1$ is a scale parameter in the production function which keeps (b_0, b_1) positive.

Figure 4 shows the phase diagram of the linear system on the invariant interval spanning the steady states (q_L^*, q_H^*) of the "low" and "high" regimes:

$$I^* = \left[q_L^*, q_H^*\right] = [b_0/(1-\alpha), b_1/(1-\alpha)] \ . \tag{28}$$

Solutions of this system are sequences of asset prices and, more generally, probability distributions of asset prices on the interval I^*. This is what we analyze immediately below.

4 Asset prices in general equilibrium

4.1 Perfect foresight

Set valued equilibrium correspondences, like the one in equation (26), typically have a large number of perfect-foresight or rational-expectations

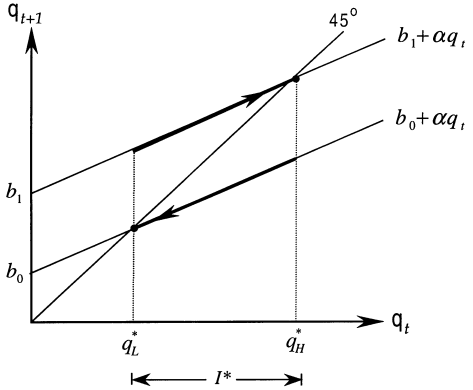

Figure 4. Dynamics of asset prices on the invariant interval

equilibria. To make some sense out of these we will need an *equilibrium selection criterion*, that is, a rule which tells us how the economy selects at each period among the two branches of the correspondence. Without any such selection rule equation (26) allows, for instance, a countable infinity of limit cycles. In fact, it is easy to show that for any pair (m, n) of non-negative integers such that $m + n \geq 1$, there is an asymptotically stable deterministic periodic cycle with period $m + n$ which keeps the economy for $m \geq 0$ periods in the high regime of increasing asset prices, followed by $n \geq 0$ periods of declining asset prices in the low regime, and then starting over. The price sequence $(q_1, q_2, \ldots, q_m, \ldots, q_{m+n})$ over this cycle is easily seen to satisfy

$$q_t = \begin{cases} q_H^* - \frac{1-\alpha^n}{1-\alpha^{m+n}}\left(q_H^* - q_L^*\right)\alpha^t, & \text{for } t = 1, \ldots, m \\ q_L^* + \frac{1-\alpha^m}{1-\alpha^{m+n}}\left(q_H^* - q_L^*\right)\alpha^{t-m}, & \text{for } t = m+1, \ldots, m+n \end{cases} \tag{29}$$

As a check on these formulas, note that the case $(m, n) = (1, 0)$ yields the high steady state $q_t = q_H^*$, for all t, while the case $(m, n) = (0, 1)$ yields the low state $q_t = q_L^*$ for all t.

4.2 Selecting an equilibrium: history versus noise trading

Selecting an equilibrium among the infinitely many deterministic and stochastic solutions to equation (26) means that we must describe a stochastic process x_t on the binary set $\{L, H\}$ which chooses in each period the low or high branch of the equilibrium correspondence. This procedure may obey a set of "reasonable axioms", a common practice in game theory (see Fudenberg and Tirole, 1995), or else summarize some form of history-dependent learning (see Evans and Honkapohja, 1998).

Whatever selection process we choose will not affect the economic structure of the model; the parameters (α, q_L^*, q_H^*) in equation (26) and Figure 4 are independent of the probability distribution of asset prices because the logarithmic utility and endowment structure we are using keep the saving decisions of individuals independent of interest rates. In the sequel we focus on an intuitively appealing, but otherwise arbitrary, class of equilibrium selection processes defined by two transition functions

$$\pi_H : [0,1] \to [0,1]; \quad \pi_L : [0,1] \to [0,1] \ , \tag{30}$$

mapping the transformed state variable

$$m_t = \frac{q_t - q_L^*}{q_H^* - q_L^*} \in [0,1] \tag{31}$$

into the probabilities of selecting a particular branch of the equilibrium correspondence. Specifically, we define the conditional transition probabilities

$$\pi_H(m) = \Pr\{x_{t+1} = H \mid x_t = H, m_t = m\} \tag{32}$$

$$\pi_L(m) = \Pr\{x_{t+1} = L \mid x_t = L, m_t = m\} \tag{33}$$

which describe the likelihood of selecting the same type of equilibrium on two successive dates, conditional on the current value of the state variable.

It makes intuitive sense to think of (π_L, π_H) as outcomes of an unspecified learning process that depends on history. For example, π_H may be an increasing function if selection displays persistence, a decreasing function if it displays mean-reversion, or else be a constant if selection is completely random, as in much of the "sunspots" literature. The functions (π_H, π_L) select equilibria from the following form of equation (26)

$$m_{t+1} = \begin{cases} \alpha m_t, & \text{if } x_{t+1} = L \\ 1 - \alpha + \alpha m_t, & \text{if } x_{t+1} = H \end{cases} \tag{34}$$

in which m_t is defined by equation (31).

A deterministic selection example is to choose a critical value $\overline{m}$ at which both regimes change, i.e.,

$$\begin{aligned} \pi_H(m) =& 1, \quad \pi_L(m) = 0, \quad \text{if } m \in [\overline{m}, 1] \\ \pi_H(m) =& 0, \quad \pi_L(m) = 1, \quad \text{otherwise.} \end{aligned} \tag{35}$$

Asset prices in this case are unique and deterministic, converging monotonically to the steady state $m = 0$ (which means $p^K = \exp(q_L^*)$) if $m_1 < \overline{m}$, and to the steady state $m = 1$ (which means $p^K = \exp(q_H^*)$) of $m_1 > \overline{m}$.

A more realistic selection process allows random events that capture the market influence of new or inexperienced traders, much as "noise trading" does in financial theory (for instance, Campbell and Kyle, 1993). At each point in time we now have two conditional probability distributions of asset prices, one for each current state $(x_t, m_t) = (x, m)$. If $x_t = L$, then tomorrow's state is

$$(x_{t+1}, m_{t+1}) = \begin{cases} (L, \alpha m), & \text{with probability } \pi_L(m) \\ (H, 1 - \alpha + \alpha m), & \text{with probability } 1 - \pi_L(m). \end{cases} \tag{36}$$

Alternatively, if $x_t = H$, then

$$(x_{t+1}, m_{t+1}) = \begin{cases} (L, \alpha m), & \text{with probability } 1 - \pi_H(m) \\ (H, 1 - \alpha + \alpha m), & \text{with probability } \pi_H(m)\ . \end{cases} \tag{37}$$

4.3 Measures of price volatility

Conditional on the current state (x, m), the mean and variance of one-period ahead asset prices are

$$\mu(L, m) = \alpha m + (1 - \alpha)[1 - \pi_L(m)] \tag{38}$$

$$\mu(H, m) = \alpha m + (1 - \alpha)\pi_H(m) \tag{39}$$

$$\sigma^2(L, m) = (1 - \alpha)^2 \pi_L(m)[1 - \pi_L(m)] \tag{40}$$

$$\sigma^2(H, m) = (1 - \alpha)^2 \pi_H(m)[1 - \pi_H(m)] \tag{41}$$

Conditional on the current state (x, m), a measure of volatility can be derived from the mean and variance of one-period ahead *level* of asset prices. We measure price volatility with the one-period-ahead *coefficient of variation* of p_{t+1}^K, which corresponds to one-period ahead proportional changes in asset prices. If (μ, σ) are the mean and standard deviation of log price and (M, Σ) are the corresponding magnitudes for level price, we exploit the approximations

$$\mu = \log M - \frac{1}{2}\left(\frac{\Sigma}{M}\right)^2$$
$$M = (1 + \sigma^2/2)\exp(\mu)$$

to obtain the volatility,

$$v_x^2 = \Sigma^2/M^2 = 2\log(1 + \sigma^2/2) \simeq \sigma_x^2 \ ,$$

where σ_x is the conditional one-period ahead standard deviation of q. Using (40) and (41), this implies

$$v_x \simeq \sigma_x = (1 - \alpha)[\pi(x)(1 - \pi(x))]^{1/2} \ . \tag{42}$$

All short-term asset price volatility is related to randomness in the marginal product of capital and in output per worker. But how much of actual volatility does this explain?

One way to obtain an estimate is to calculate the elasticity of the net return to capital, $f'(k) - \delta$, with respect to output, $f(k)$. A quick computation shows this elasticity to be

$$\begin{aligned} \varepsilon_r &= \frac{[-f''(k)/f'(k)]f(k)}{f'(k) - \delta} \\ &= \frac{(1 - \alpha)/\sigma}{\alpha - \delta(k/y)} \ , \end{aligned} \tag{43}$$

where α is capital's share of output, σ is the capital-labor elasticity of substitution, and δ is the depreciation rate. Calibrating this to quarterly data, we choose parameter values $(\alpha, \delta, \sigma, k/y) = (0.36, 0.02, 0.80, 10)$ and obtain

$$\varepsilon_r \approx 5 \ . \tag{44}$$

This value means that quarterly asset returns are five times as volatile as real output, a finding in accord with U.S. data reported in section 1.

5 Conclusion and extensions

We have explored a nonconvex model economy in which output and asset prices fluctuate endogenously in response to changes in the unit costs of financial intermediation. This economy displays "excess volatility" in the macroeconomic definition of the term but not in the finance definition: capital asset prices always equal the present value of dividends, even though changes in dividends/interest rates are driven exclusively by consumer beliefs.

One remaining issue is whether costs of financial intermediation, monitoring or bankruptcy, as reported in Guffey and Moore (1991) and elsewhere, support the increasing returns we need to generate volatility in this model. And if they do, will the volatility be consistent with the actual variability of U.S. output? If it is not, we may need to add fundamental shocks to this model. Another issue is how volatility is affected by an active monetary or fiscal policy. We do not treat this issue here but refer the interested reader to related work by Azariadis and Smith (1996).

Answering the original volatility paradox uncovered by Shiller, LeRoy and Porter still eludes us. What amplifies small innovations in the present value of dividends to big ones in capital asset prices? We are not convinced that the answer lies in the behavior of unsophisticated noise traders. This issue seems likely to occupy center stage in macroeconomics and finance for a while.

Appendix A: The optimum loan contract

A typical investor's payoff from the contract δ is

$$U = \begin{cases} 0, & \text{if} \quad \theta \le x, \\ b(\theta\rho - R^L) = b\rho(\theta - x), & \text{if} \quad \theta > x \end{cases} .$$

This gives the following expected payoff as a function of the loan contract

$$U(\delta) = b\rho[1 - \mu(x)], \tag{A.1}$$

where $\mu(x) = x[1 - H(x)] + \int_{\underline{\theta}}^{x} \theta dH$. Similarly, if Γ is the verification cost that a bank can potentially incur on each loan, its payoff function is given by

$$\pi^B = \begin{cases} b(\rho\theta - \rho\Gamma - R^D), & \text{if} \quad \theta \le x \\ b(\rho x - R^D), & \text{if} \quad \theta > x \end{cases} .$$

The verification cost is assumed to be proportional to the size of the loan, e.g., $\Gamma = \gamma b$, where $\gamma \in [0, \underline{\theta}]$. Let us define $q \equiv R^D/\rho$ and let

$$M(x, \gamma) \equiv x[1 - H(x)] + \int_{\underline{\theta}}^{x} (\theta - \gamma) dH \tag{A.2}$$

Expected bank profit is then[10],

$$\Pi(\delta, q) \equiv \rho b[M(x, \gamma) - q] . \tag{A.3}$$

Definition *Given* (q, γ) *and* U_0, *the loan contract* $\hat{\delta} = (b, \hat{x}, R^L)$ *is optimal if:*

1 $R^L = \rho x$, *and*
2 $\hat{x} = \arg\max \Pi(\delta, q)$ *subject to* $U(\delta) \ge U_0$.

Suppose now that the likelihood function $\mathscr{L}(\theta) = h(\theta)/[1 - H(\theta)]$, defined for $\theta \in [\underline{\theta}, \overline{\theta}]$, is increasing in θ. Then the critical state $\hat{x}$ defined above satisfies

$$\hat{x} = \arg\max_{x \in \Theta} M(x, \gamma) \tag{A.4}$$

[10] If the last term in (A.2) is negative, the loan contract is not renegotiation proof. The bank would throw away a bankrupt project rather than monitor it; it should then be possible to change (b, x) and improve the contract by lowering x instead of throwing away θb worth of resources. To ensure that bankrupt projects are not abandoned we assume γ is "small", ie, $\gamma \in [0, \underline{\theta}]$ which guarantees $\int_{\underline{\theta}}^{x} (\theta - \gamma) dH \ge 0$.

Thus, $\hat{x}$ solves the equation

$$M_x(x,\gamma) = [1 - H(x)][1 - \gamma \mathscr{L}(x)] \leq 0, \quad \text{and} = 0 \quad \text{if } x > \underline{\theta} \ . \tag{A.5}$$

Since $\Gamma = \gamma b$, $M(\underline{\theta},\gamma) = \underline{\theta}$ and $M(\overline{\theta},\gamma) = 1 - \gamma$, there exists a critical value of γ,

$$\gamma_c = \min\{1, 1/h(\underline{\theta})\} \ ,$$

and a weakly decreasing function of γ, $\hat{x} : [0,1] \to \Theta$, such that

$$\hat{x}(0) = \overline{\theta}, \hat{x}(\gamma) = \underline{\theta} \ \ \forall \, \gamma \in [\gamma_c, 1] \ . \tag{A.6}$$

In equilibrium bank profits must be driven to zero if there is free entry into banking so that $\Pi(\delta, q) = 0$. From (A.3), the exogenous price q must satisfy

$$q = \begin{cases} M(\hat{x},\gamma) & \text{if} \quad \gamma \in [0,\gamma_c], \\ \underline{\theta} & \text{if} \quad \gamma \in [\gamma_c, 1]. \end{cases} \ . \tag{A.7}$$

The following theorem describes completely the optimum contract.

Theorem *Given* γ *and* (*A.6*), *the optimum loan contract* $\hat{\delta} = (\hat{b}, \hat{x}, \hat{R}^L)$ *satisfies equation* (*A.4*) *and* $U = U_0$, *i.e.,* $\hat{b} = U_0/\rho[1 - \mu(\hat{x}(\gamma))]$. *In particular,* $\gamma \in (\gamma_c, 1]$ *implies no state verification and no interest rate spread* $(x = \underline{\theta}, R^L = R^D)$, *while* $\gamma \in [0, \gamma_c)$ *implies verification in some states and a positive interest rate spread* $(x > \underline{\theta}, R^L > R^D)$.

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